Collapse of the mean curvature flow for certain kind of invariant hypersurfaces in a Hilbert space

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Abstract

In this paper, we investigate the regularized mean curvature flow starting from an invariant hypersurface in a Hilbert space equipped with an isometric and almost free action of a Hilbert Lie group whose orbits are regularized minimal. We prove that, if the invariant hypersurface satisfies a certain kind of horizontally convexity condition, then it collapses to an orbit of the Hilbert Lie group action along the regularized mean curvature flow.

1 Introduction

In this paper, we consider the regularized mean curvature flow in a (separable infinite dimensional) Hilbert space V. Let M be a Hilbert manifold and f_t $(0 \leq t < T)$ be a C^∞-family of immersions of finite codimension of M into V. Assume that each f_t is regularizable, where "regularizability" means that f_t is proper Fredholm and that, for each normal vector v of M, the regularized trace $\mathrm{Tr}_r\,(A_t)_v$ of the shape operator $(A_t)_v$ of f_t and the trace $\mathrm{Tr}\,(A_t)_v^2$ of $(A_t)_v^2$ exist. Then each shape operator $(A_t)_v$ is a compact operator. Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0,T) \to V$ by $F(x,t) := f_t(x)$ $((x,t) \in M \times [0,T))$. We call f_t's $(0 \leq t < T)$ the *regularized mean curvature flow* if the following evolution equation holds:

$$\frac{\partial F}{\partial t} = \triangle^r_t f_t. \tag{1.1}$$

Here $\triangle^r_t f_t$ is defined as the vector field along f_t satisfying

$$\langle \triangle^r_t f_t, v \rangle := \mathrm{Tr}_r \langle (\nabla^t df_t)(\cdot,\cdot), v\rangle^\sharp \quad (\forall\, v \in V),$$

where ∇^t is the Riemannian connection of the metric g_t on M induced from the metric $\langle\ ,\ \rangle$ of V by f_t, $\langle(\nabla^t df_t)(\cdot,\cdot),v\rangle^\sharp$ is the $(1,1)$-tensor field on M defined by $g_t(\langle(\nabla^t df_t)(\cdot,\cdot),v\rangle^\sharp(X),Y)=\langle(\nabla^t df_t)(X,Y),v\rangle$ $(X,Y\in TM)$ and $\mathrm{Tr}_r(\cdot)$ is the regularized trace of $(\cdot)$. Note that $\triangle^r_t f_t$ is equal to H_t. R. S. Hamilton ([Ha]) proved the existenceness and the uniqueness (in short time) of solutions of a weakly parabolic equation for sections of a finite dimensional vector bundle. The evolution equation (1.1) is regarded as the evolution equation for sections of the *infinite* dimensional vector bundle $M\times V$ over M. Also, M is not compact. Hence we cannot apply the Hamilton's result to this evolution equation (1.1). Also, we must impose certain kind of infinite dimensional invariantness for f because M is not compact. Thus, we cannot show the existenceness and the uniqueness (in short time) of solutions of (1.1) in general. However we ([K]) showed the existenceness and the uniqueness (in short time) of solutions of (1.1) in the following special case. We consider a isometric almost free action of a Hilbert Lie group G on a Hilbert space V whose orbits are regularized minimal, that is, they are regularizable submanifold and their regularized mean curvature vectors vanish, where "almost free" means that the isotropy group of the action at each point is finite. Let $M(\subset V)$ be a G-invariant submanifold in V. Assume that the image of M by the orbit map of the G-action is compact. Let f be the inclusion map of M into V. Then we showed that the regularized mean curvature flow starting from M exists uniquely in short time. In this paper, we consider the case where M is a hypersurface. The purpose of this paper is to prove that M collapses to an orbit of the Hilbert Lie group action along the regularized mean curvature flow when it satisfies a certain kind of horizontally strongly convexity condition and horizontally volume condition (see Theorem 6.1).

2 The regularized mean curvature flow

Let f be an immersion of an (infinite dimensional) Hilbert manifold M into a Hilbert space V and A the shape tensor of f. If $\mathrm{codim}\,M<\infty$, if the differential of the normal exponential map $\exp^\perp$ of f at each point of M is a Fredholm operator and if the restriction $\exp^\perp$ to the unit normal ball bundle of f is proper, then M is called a *proper Fredholm submanifold*. In this paper, we then call f a *proper Fredholm immersion*. Furthermore, if, for each normal vector v of M, the regularized trace $\mathrm{Tr}_r\,A_v$ and $\mathrm{Tr}\,A_v^2$ exist, then M is called *regularizable submanifold*, where $\mathrm{Tr}_r\,A_v$ is defined by $\mathrm{Tr}_r\,A_v:=\sum\limits_{i=1}^\infty(\mu_i^++\mu_i^-)$ $(\mu_1^-\le\mu_2^-\le\cdots\le 0\le\cdots\le\mu_2^+\le\mu_1^+$: the spectrum of $A_v)$. In this paper, we then call f *regularizable immersion*. If f is a regulalizable immersion, then the *regularized mean curvature vector* H of f is defined by $\langle H,v\rangle=\mathrm{Tr}_r\,A_v$ $(\forall\,v\in T^\perp M)$, where $\langle\ ,\ \rangle$ is the inner product of V and $T^\perp M$

is the normal bundle of f. If $H = 0$, then f is said to be *minimal*. In particular, if f is of codimension one, then we call the norm $||H||$ of H the *regularized mean curvature function* of f.

Let f_t $(0 \leq t < T)$ be a C^∞-family of regularizable immersions of M into V. Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0,T) \to V$ by $F(x,t) := f_t(x)$ $((x,t) \in M \times [0,T))$. If $\frac{\partial F}{\partial t} = H_t$ holds, then we call f_t $(0 \leq t < T)$ the *regularized mean curvature flow*.

3 Evolution equations

Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$. The orbit space V/G is a (finite dimensional) C^∞-orbifold. Let $\phi : V \to V/G$ be the orbit map and set $N := V/G$. Here we give an example of such an isometric almost free action of a Hilbert Lie group.

Example. Let G be a compact semi-simple Lie group, K a closed subgroup of G and Γ a discrete subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Assume that a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ exists. Let B be the Killing form of $\mathfrak{g}$. Give G the bi-invariant metric induced from B. Let $H^0([0,1],\mathfrak{g})$ be the Hilbert space of all paths in the Lie algebra $\mathfrak{g}$ of G which are L^2-integrable with respect to B. Also, let $H^1([0,1],G)$ the Hilbert Lie group of all paths in G which are of class H^1 with respect to g. This group $H^1([0,1],G)$ acts on $H^0([0,1],\mathfrak{g})$ isometrically and transitively as a gauge action:

$$\begin{aligned}(a * u)(t) = \mathrm{Ad}_G(a(t))(u(t)) - (R_{a(t)})_*^{-1}(a'(t))\\ (a \in H^1([0,1],G),\ u \in H^0([0,1],\mathfrak{g})),\end{aligned}$$

where Ad_G is the adjoint representation of G and $R_{a(t)}$ is the right translation by $a(t)$ and a' is the weak derivative of a. Set $P(G,\Gamma\times K) := \{a \in H^1([0,1],G) \,|\, (a(0),a(1)) \in \Gamma \times K\}$. The group $P(G,\Gamma \times K)$ acts on $H^0([0,1],\mathfrak{g})$ almost freely and isometrically, and the orbit space of this action is diffeomorphic to the orbifold $\Gamma \setminus G \,/\, K$. Furthermore, each orbit of this action is regularizable and minimal.

Give N the Riemannian orbimetric such that ϕ is a Riemannian orbisubmersion. Let $f : M \hookrightarrow V$ be a G-invariant submanifold immersion such that $(\phi\circ f)(M)$ is compact. For this immersion f, we can take an orbiimmesion $\overline{f}$ of a compact orbifold $\overline{M}$ into N and an orbifold submersion $\phi_M : M \to \overline{M}$ with $\phi\circ f = \overline{f}\circ\phi_M$. Let f_t $(0 \leq t < T)$ be the regularized mean curvature flow starting from f and $\overline{f}_t$ $(0 \leq t < T)$ the mean curvature flow starting from $\overline{f}$. The existenceness and the uniqueness of these flows

is assured by Theorem 3.1 and Proposition 4.1 in [K]. Note that $\phi\circ f_t=\overline{f}_t\circ\phi_M$ holds for all $t\in[0,T)$. Define a map $F:M\times[0,T)\to V$ by $F(u,t):=f_t(u)$ ($(u,t)\in M\times[0,T)$) and a map $\overline{F}:\overline{M}\times[0,T)\to N$ by $\overline{F}(x,t):=\overline{f}_t(x)$ ($(x,t)\in\overline{M}\times[0,T)$). Denote by H_t the regularized mean curvature vector of f_t and $\overline{H}_t$ the mean curvature vector of $\overline{f}_t$. Since ϕ has minimal regularizable fibres, H_t is the horizontal lift of $\overline{H}_t$. It is clear that $\phi\circ f_t=\overline{f}_t\circ\phi_M$ holds for all $t\in[0,T)$. Assume that the codimension of M is equal to one. Denote by $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$) the horizontal (resp. vertical) distribution of ϕ. Denote by $\mathrm{pr}_{\widetilde{\mathcal{H}}}$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}$) the orthogonal projection of TV onto $\widetilde{\mathcal{H}}$ (resp. $\widetilde{\mathcal{V}}$). For simplicity, for $X\in TV$, we denote $\mathrm{pr}_{\widetilde{\mathcal{H}}}(X)$ (resp. $\mathrm{pr}_{\widetilde{\mathcal{V}}}(X)$) by $X_{\widetilde{\mathcal{H}}}$ (resp. $X_{\widetilde{\mathcal{V}}}$). Define a distribution $\mathcal{H}_t$ on M by $f_{t*}((\mathcal{H}_t)_u)=f_{t*}(T_uM)\cap\widetilde{\mathcal{H}}_{f_t(u)}$ ($u\in M$) and a distribution $\mathcal{V}_t$ on M by $f_{t*}((\mathcal{V}_t)_u)=\widetilde{\mathcal{V}}_{f_t(u)}$ ($u\in M$). Note that $\mathcal{V}_t$ is independent of the choice of $t\in[0,T)$. Denote by g_t,h_t,A_t,H_t and ξ_t the induced metric, the second fundamental form, the shape tensor and the regularized mean curvature vector and the unit normal vector field of f_t, respectively. The group G acts on M through f_t. Since $\phi:V\to V/G$ is a G-orbibundle and $\widetilde{\mathcal{H}}$ is a connection of the orbibundle, it follows from Proposition 4.1 that this action $G\curvearrowright M$ is independent of the choice of $t\in[0,T)$. It is clear that quantities g_t,h_t,A_t and H_t are G-invariant. Also, let ∇^t be the Riemannian connection of g_t. Let π_M be the projection of $M\times[0,T)$ onto M. For a vector bundle E over M, denote by π_M^*E the induced bundle of E by π_M. Also denote by $\Gamma(E)$ the space of all sections of E. Define a section g of $\pi_M^*(T^{(0,2)}M)$ by $g(u,t)=(g_t)_u$ ($(u,t)\in M\times[0,T)$), where $T^{(0,2)}M$ is the $(0,2)$-tensor bundle of M. Similarly, we define a section h of $\pi_M^*(T^{(0,2)}M)$, a section A of $\pi_M^*(T^{(1,1)}M)$, a map $H:M\times[0,T)\to TV$ and a map $\xi:M\times[0,T)\to TV$. We regard H and ξ as V-valued functions over $M\times[0,T)$ under the identification of T_uV's ($u\in V$) and V. Define a subbundle $\mathcal{H}$ (resp. $\mathcal{V}$) of π_M^*TM by $\mathcal{H}_{(u,t)}:=(\mathcal{H}_t)_u$ (resp. $\mathcal{V}_{(u,t)}:=(\mathcal{V}_t)_u$). Denote by $\mathrm{pr}_{\mathcal{H}}$ (resp. $\mathrm{pr}_{\mathcal{V}}$) the orthogonal projection of $\pi_M^*(TM)$ onto $\mathcal{H}$ (resp. $\mathcal{V}$). For simplicity, for $X\in\pi_M^*(TM)$, we denote $\mathrm{pr}_{\mathcal{H}}(X)$ (resp. $\mathrm{pr}_{\mathcal{V}}(X)$) by $X_{\mathcal{H}}$ (resp. $X_{\mathcal{V}}$). The bundle $\pi_M^*(TM)$ is regarded as a subbundle of $T(M\times[0,T))$. For a section B of $\pi_M^*(T^{(r,s)}M)$, we define $\dfrac{\partial B}{\partial t}$ by $\left(\dfrac{\partial B}{\partial t}\right)_{(u,t)}:=\dfrac{dB_{(u,t)}}{dt}$, where the right-hand side of this relation is the derivative of the vector-valued function $t\mapsto B_{(u,t)}$ ($\in T_u^{(r,s)}M$). Also, we define a section $B_{\mathcal{H}}$ of $\pi_M^*(T^{(r,s)}M)$ by

$$B_{\mathcal{H}}=(\underbrace{\mathrm{pr}_{\mathcal{H}}\otimes\cdots\otimes\mathrm{pr}_{\mathcal{H}}}_{(r-\text{times})})\circ B\circ(\underbrace{\mathrm{pr}_{\mathcal{H}}\otimes\cdots\otimes\mathrm{pr}_{\mathcal{H}}}_{(s-\text{times})}).$$

The restriction of $B_{\mathcal{H}}$ to $\mathcal{H}\times\cdots\times\mathcal{H}$ (s-times) is regarded as a section of the (r,s)-tensor bundle $\mathcal{H}^{(r,s)}$ of $\mathcal{H}$. This restriction also is denoted by the same symbol $B_{\mathcal{H}}$. Let D_M (resp. $D_{[0,T)}$) be the subbundle of $T(M\times[0,T))$ defined by $(D_M)_{(u,t)}:=T_{(u,t)}(M\times\{t\})$ (resp. $(D_{[0,T)})_{(u,t)}:=T_{(x,t)}(\{u\}\times[0,T))$ for each

$(u,t) \in M \times [0,T)$. Denote by $v^L_{(u,t)}$ the horozontal lift of $v \in T_uM$ to (u,t) with respect to π_M, where we note that $v^L_{(u,t)} \in (D_M)_{(u,t)}$. Under the identification of $((u,t),v)(=v) \in (\pi^*TM)_{(u,t)}$ with $v^L_{(u,t)} \in (D_M)_{(u,t)}$, we identify π_M^*TM with D_M. For a tangent vector field X on M (or an open set U of M), we define $\overline{X} \in \Gamma(\pi_M^*TM)(=\Gamma(D_M))$ (or $\Gamma((\pi_M^*TM)|_U)(=\Gamma((D_M)|_U))$) by $\overline{X}_{(u,t)} := ((u,t),X_u)(=(X_u)^L_{(u,t)})$ $((u,t) \in M \times [0,T))$. Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Let ∇ be the connection of π_M^*TM defined by

$$(\nabla_X Y)_{(u,t)} := \nabla^t_{X_{(u,t)}} Y_{(\cdot,t)} \ \text{ and } \ (\nabla_{\frac{\partial}{\partial t}} Y)_{(u,t)} := \frac{dY_{(u,\cdot)}}{dt}$$

for $X \in \Gamma(D_M)$ and $Y \in \Gamma(\pi_M^*TM)$, where we regard as $X_{(u,t)} \in T_uM$, $Y_{(\cdot,t)} \in \Gamma(TM)$ and $Y_{(u,\cdot)} \in C^\infty([0,T),T_uM)$. Note that $\nabla_{\frac{\partial}{\partial t}}\overline{X} = 0$. Denote by the same symbol ∇ the connection of $\pi_M^*T^{(r,s)}M$ defined in terms of ∇^t's similarly. Define a connection $\nabla^{\mathcal{H}}$ of $\mathcal{H}$ by $\nabla^{\mathcal{H}}_X Y := (\nabla_X Y)_{\mathcal{H}}$ for $X \in \Gamma(M \times [0,T))$ and $Y \in \Gamma(\mathcal{H})$. Similarly, define a connection $\nabla^{\mathcal{V}}$ of $\mathcal{V}$ by $\nabla^{\mathcal{V}}_X Y := (\nabla_X Y)_{\mathcal{V}}$ for $X \in \Gamma(M \times [0,T))$ and $Y \in \Gamma(\mathcal{V})$. Now we shall derive the evolution equations for some geometric quantities. In [K], we derived the following evolution equations.

Lemma 3.1. *The sections $(g_{\mathcal{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfy the following evolution equation:*

$$\frac{\partial g_{\mathcal{H}}}{\partial t} = -2||H||h_{\mathcal{H}},$$

where $||H|| := \sqrt{\langle H,H\rangle}$.

Lemma 3.2. *The unit normal vector fields ξ_t's satisfy the following evolution equation:*

$$\frac{\partial \xi}{\partial t} = -F_*(\mathrm{grad}_g||H||),$$

where $\mathrm{grad}_g(||H||)$ is the element of $\pi_M^*(TM)$ such that $d||H||(X) = g(\mathrm{grad}_g||H||,X)$ for any $X \in \pi_M^*(TM)$.

Let S_t $(0 \le t < T)$ be a C^∞-family of a (r,s)-tensor fields on M and S a section of $\pi_M^*(T^{(r,s)}M)$ defined by $S_{(u,t)} := (S_t)_u$. We define a section $\triangle_{\mathcal{H}}S$ of $\pi_M^*(T^{(r,s)}M)$ by

$$(\triangle_{\mathcal{H}}S)_{(u,t)} := \sum_{i=1}^n \nabla_{e_i}\nabla_{e_i}S,$$

where ∇ is the connection of $\pi_M^*(T^{(r,s)}M)$ (or $\pi_M^*(T^{(r,s+1)}M)$) induced from ∇ and $\{e_1,\cdots,e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$. Also, we

define a section $\bar{\triangle}_{\mathcal{H}}S_{\mathcal{H}}$ of $\mathcal{H}^{(r,s)}$ by

$$(\triangle^{\mathcal{H}}_{\mathcal{H}}S_{\mathcal{H}})_{(u,t)} := \sum_{i=1}^{n} \nabla^{\mathcal{H}}_{e_i}\nabla^{\mathcal{H}}_{e_i}S_{\mathcal{H}},$$

where $\nabla^{\mathcal{H}}$ is the connection of $\mathcal{H}^{(r,s)}$ (or $\mathcal{H}^{(r,s+1)}$) induced from $\nabla^{\mathcal{H}}$ and $\{e_1,\cdots,e_n\}$ is as above. Let $\mathcal{A}^{\phi}$ be the section of $T^*V\otimes T^*V\otimes TV$ defined by

$$\mathcal{A}^{\phi}_X Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}}Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{H}}}}Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X,Y\in TV).$$

Also, let $\mathcal{T}^{\phi}$ be the section of $T^*V\otimes T^*V\otimes TV$ defined by

$$\mathcal{T}^{\phi}_X Y := (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}}Y_{\widetilde{\mathcal{H}}})_{\widetilde{\mathcal{V}}} + (\widetilde{\nabla}_{X_{\widetilde{\mathcal{V}}}}Y_{\widetilde{\mathcal{V}}})_{\widetilde{\mathcal{H}}} \quad (X,Y\in TV).$$

Also, let $\mathcal{A}_t$ be the section of $T^*M\otimes T^*M\otimes TM$ defined by

$$(\mathcal{A}_t)_X Y := (\nabla^t_{X_{\mathcal{H}_t}}Y_{\mathcal{H}_t})_{\mathcal{V}_t} + (\nabla^t_{X_{\mathcal{H}_t}}Y_{\mathcal{V}_t})_{\mathcal{H}_t} \quad (X,Y\in TM).$$

Also let $\mathcal{A}$ be the section of $\pi_M^*(T^*M\otimes T^*M\otimes TM)$ defined in terms of $\mathcal{A}_t$'s ($t\in[0,T)$). Also, let $\mathcal{T}_t$ be the section of $T^*M\otimes T^*M\otimes TM$ defined by

$$(\mathcal{T}_t)_X Y := (\nabla^t_{X_{\mathcal{V}_t}}Y_{\mathcal{V}_t})_{\mathcal{H}_t} + (\nabla^t_{X_{\mathcal{V}_t}}Y_{\mathcal{H}_t})_{\mathcal{V}_t} \quad (X,Y\in TM).$$

Also let $\mathcal{T}$ be the section of $\pi_M^*(T^*M\otimes T^*M\otimes TM)$ defined in terms of $\mathcal{T}_t$'s ($t\in[0,T)$). Clearly we have

$$F_*(\mathcal{A}_X Y) = \mathcal{A}^{\phi}_{F_*X}F_*Y$$

for $X,Y\in\mathcal{H}$ and

$$F_*(\mathcal{T}_W X) = \mathcal{T}^{\phi}_{F_*W}F_*X$$

for $X\in\mathcal{H}$ and $W\in\mathcal{V}$. Let E be a vector bundle over M. For a section S of $\pi_M^*(T^{(0,r)}M\otimes E)$, we define $\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots)$ by

$$(\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots))_{(u,t)} = \sum_{i=1}^{n} S_{(u,t)}(\cdots,\overset{j}{e_i},\cdots,\overset{k}{e_i},\cdots)$$

($(u,t)\in M\times[0,T)$), where $\{e_1,\cdots,e_n\}$ is an orthonormal base of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$, $S(\cdots,\overset{j}{\bullet},\cdots,\overset{k}{\bullet},\cdots)$ means that $\bullet$ is entried into the j-th component and the k-th component of S and $S_{(u,t)}(\cdots,\overset{j}{e_i},\cdots,\overset{k}{e_i},\cdots)$ means that e_i is entried into the j-th component and the k-th component of $S_{(u,t)}$.

In [K], we derived the following relation.

Lemma 3.3. *Let S be a section of $\pi_M^*(T^{(0,2)}M)$ which is symmetric with respect to g. Then we have*

$$\begin{aligned}(\triangle_{\mathcal{H}}S)_{\mathcal{H}}(X,Y) = &(\triangle^{\mathcal{H}}_{\mathcal{H}}S_{\mathcal{H}})(X,Y)\\ &-2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}S)(\mathcal{A}_{\bullet}X,Y)) - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}((\nabla_{\bullet}S)(\mathcal{A}_{\bullet}Y,X))\\ &-\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}X),Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}Y),X)\\ &-\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S((\nabla_{\bullet}\mathcal{A})_{\bullet}X,Y) - \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S((\nabla_{\bullet}\mathcal{A})_{\bullet}Y,X)\\ &-2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}S(\mathcal{A}_{\bullet}X,\mathcal{A}_{\bullet}Y)\end{aligned}$$

for $X,Y\in\mathcal{H}$, where ∇ is the connection of $\pi_M^*(T^{(1,2)}M)$ induced from ∇.

Also we derived the following Simons-type identity.

Lemma 3.4. *We have*

$$\triangle_{\mathcal{H}}h = \nabla d||H|| + ||H||(A^2)_{\sharp} - (\mathrm{Tr}\,(A^2)_{\mathcal{H}})h,$$

where $(A^2)_{\sharp}$ is the element of $\Gamma(\pi_M^*T^{(0,2)}M)$ defined by $(A^2)_{\sharp}(X,Y) := g(A^2X,Y)$ $(X,Y\in\pi_M^*TM)$.

Note. In the sequel, we omit the notation F_* for simplicity.

Define a section $\mathcal{R}$ of $\pi_M^*(\mathcal{H}^{(0,2)})$ by

$$\begin{aligned}\mathcal{R}(X,Y) := &\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}X),Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}(\mathcal{A}_{\bullet}Y),X)\\ &+\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\nabla_{\bullet}\mathcal{A})_{\bullet}X,Y) + \mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\nabla_{\bullet}\mathcal{A})_{\bullet}Y,X)\\ &+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}(\nabla_{\bullet}h)(\mathcal{A}_{\bullet}X,Y) + 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}(\nabla_{\bullet}h)(\mathcal{A}_{\bullet}Y,X)\\ &+2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(\mathcal{A}_{\bullet}X,\mathcal{A}_{\bullet}Y) \qquad\qquad (X,Y\in\mathcal{H}).\end{aligned}$$

From Lemmas 3.2, 3.3 and 3.4, we ([K]) derived the following evolution equation for $(h_{\mathcal{H}})_t$'s.

Lemma 3.5. *The sections $(h_{\mathcal{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfies the following evolution equation:*

$$\begin{aligned}\frac{\partial h_{\mathcal{H}}}{\partial t}(X,Y) = &(\triangle^{\mathcal{H}}_{\mathcal{H}}h_{\mathcal{H}})(X,Y) - 2||H||((A_{\mathcal{H}})^2)_{\sharp}(X,Y) - 2||H||((\mathcal{A}^{\phi}_{\xi})^2)_{\sharp}(X,Y)\\ &+\mathrm{Tr}\left((A_{\mathcal{H}})^2 - ((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\right)h_{\mathcal{H}}(X,Y) - \mathcal{R}(X,Y)\end{aligned}$$

for $X,Y\in\mathcal{H}$.

From Lemma 3.1, we ([K]) derived the following relation.

Lemma 3.6. *Let X and Y be local sections of $\mathcal{H}$ such that $g(X,Y)$ is constant. Then we have $g(\nabla_{\frac{\partial}{\partial t}}X,Y)+g(X,\nabla_{\frac{\partial}{\partial t}}Y)=2||H||h(X,Y)$.*

Also, we ([K]) derived the following relations for $\mathcal{R}$.

Lemma 3.7. *For $X,Y\in\mathcal{H}$, we have*

$$\begin{aligned}\mathcal{R}(X,Y)=&\,2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\left(\langle(\mathcal{A}^\phi_\bullet X,\mathcal{A}^\phi_\bullet(A_{\mathcal{H}}Y)\rangle+\langle(\mathcal{A}^\phi_\bullet Y,\mathcal{A}^\phi_\bullet(A_{\mathcal{H}}X)\rangle\right)\\ &+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\left(\langle(\mathcal{A}^\phi_\bullet X,\mathcal{A}^\phi_Y(A_{\mathcal{H}}\bullet)\rangle+\langle(\mathcal{A}^\phi_\bullet Y,\mathcal{A}^\phi_X(A_{\mathcal{H}}\bullet)\rangle\right)\\ &+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\left(\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\xi Y,\mathcal{A}^\phi_\bullet X\rangle+\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\xi X,\mathcal{A}^\phi_\bullet Y\rangle\right)\\ &+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\left(\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\bullet X,\mathcal{A}^\phi_\xi Y\rangle+\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\bullet Y,\mathcal{A}^\phi_\xi X\rangle\right)\\ &+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{T}^\phi_{\mathcal{A}^\phi_\bullet X}\xi,\mathcal{A}^\phi_\bullet Y\rangle,\end{aligned}\tag{3.1}$$

where we omit F_*. In particular, we have

$$\begin{aligned}\mathcal{R}(X,X)=&\,4\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}^\phi_\bullet X,\mathcal{A}^\phi_\bullet(A_{\mathcal{H}}X)\rangle+4\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}^\phi_\bullet X,\mathcal{A}^\phi_X(A_{\mathcal{H}}\bullet)\rangle\\ &+3\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\xi X,\mathcal{A}^\phi_\bullet X\rangle+2\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle(\widetilde{\nabla}_\bullet\mathcal{A}^\phi)_\bullet X,\mathcal{A}^\phi_\xi X\rangle\\ &+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\langle\mathcal{A}^\phi_\bullet X,(\widetilde{\nabla}_X\mathcal{A}^\phi)_\xi\bullet\rangle\end{aligned}$$

and hence

$$\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(\bullet,\bullet)=0.$$

Simple proof of the third relation. We give a simple proof of $\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(\bullet,\bullet)=0$. Take any $(u,t)\in M\times[0,T)$ and an orthonormal base $(e_1,\cdots,e_n)$ of $\mathcal{H}_{(u,t)}$ with respect to $g_{(u,t)}$. According to Lemma 3.3 and the definiton of $\mathcal{R}$, we have

$$\begin{aligned}&(\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(\bullet,\bullet))_{(u,t)}=(\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\triangle_{\mathcal{H}}h)_{\mathcal{H}}(\bullet,\bullet))_{(u,t)}-(\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\triangle^{\mathcal{H}}_{\mathcal{H}}h_{\mathcal{H}})(\bullet,\bullet))_{(u,t)}\\ &=(\triangle_{\mathcal{H}}||H||)_{(u,t)}-(\triangle^{\mathcal{H}}_{\mathcal{H}}||H||)_{(u,t)}=\sum_{i=1}^n\left((\nabla d||H||)(e_i,e_i)-(\nabla^{\mathcal{H}}d||H||)(e_i,e_i)\right)\\ &=-\sum_{i=1}^n(\mathcal{A}_{e_i}e_i)||H||=0,\end{aligned}$$

where we use $||H||=\sum_{i=1}^n h(e_i,e_i)$ (which holds because the fibres of ϕ is regularized minimal). q.e.d.

Also, we ([K]) derived the following evolution equation for $||H_t||$'s.

Lemma 3.8. *The norms $||H_t||$'s of H_t satisfy the following evolution equation:*

$$\frac{\partial ||H||}{\partial t} = \triangle_{\mathcal{H}}||H|| + ||H||||A_{\mathcal{H}}||^2 - 3||H||\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}.$$

Also, we ([K]) derived the following evolution equation for $||(A_{\mathcal{H}})_t||^2$.

Lemma 3.9. *The quantities $||(A_{\mathcal{H}})_t||^2$'s satisfy the following evolution equation:*

$$\begin{aligned}\frac{\partial ||A_{\mathcal{H}}||^2}{\partial t} = & \triangle_{\mathcal{H}}(||A_{\mathcal{H}}||^2) - 2||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2 \\ & +2||A_{\mathcal{H}}||^2 \left(||A_{\mathcal{H}}||^2 - \mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\right) \\ & -4||H||\mathrm{Tr}\left(((\mathcal{A}^{\phi}_{\xi})^2) \circ A_{\mathcal{H}}\right) - 2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet, \bullet).\end{aligned}$$

Also, we ([K]) derived the following evolution equation.

Lemma 3.10. *The quantities $||(A_{\mathcal{H}})_t||^2 - \frac{||H_t||^2}{n}$'s satisfy the following evolution equation:*

$$\begin{aligned}\frac{\partial(||(A_{\mathcal{H}})_t||^2 - \frac{||H||^2}{n})}{\partial t} = & \triangle_{\mathcal{H}}\left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n}\right) + \frac{2}{n}||\mathrm{grad}||H||||^2 \\ & +2||A_{\mathcal{H}}||^2 \times \left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n}\right) - 2||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2 \\ & -2\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \times \left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n}\right) \\ & -4||H||\left(\mathrm{Tr}\left((\mathcal{A}^{\phi}_{\xi})^2 \circ \left(A_{\mathcal{H}} - \frac{||H||}{n}\mathrm{id}\right)\right)\right) \\ & -2\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}\left(\left(A_{\mathcal{H}} - \frac{||H||}{n}\mathrm{id}\right)\bullet, \bullet\right),\end{aligned}$$

where $\mathrm{grad}||H||$ *is the gradient vector field of* $||H||$ *with respect to* g *and* $||\mathrm{grad}||H||||$ *is the norm of* $\mathrm{grad}||H||$ *with respect to* g.

Set $n := \dim \mathcal{H} = \dim \overline{M}$ and denote by $\bigwedge^n \mathcal{H}^*$ the exterior product bundle of degree n of $\mathcal{H}^*$. Let $d\mu_{g_{\mathcal{H}}}$ be the section of $\pi_M^*(\bigwedge^n \mathcal{H}^*)$ such that $(d\mu_{g_{\mathcal{H}}})_{(u,t)}$ is the volume element of $(g_{\mathcal{H}})_{(u,t)}$ for any $(u,t) \in M \times [0,T)$. Then we can derive the following evolution equation for $\{(d\mu_{g_{\mathcal{H}}})_{(\cdot,t)}\}_{t\in[0,T)}$.

Lemma 3.11. *The family* $\{(d\mu_{g_{\mathcal{H}}})_{(\cdot,t)}\}_{t\in[0,T)}$ *satisfies*

$$\frac{\partial\mu_{g_{\mathcal{H}}}}{\partial t} = -||H||^2\cdot d\mu_{g_{\mathcal{H}}}.$$

4 A maximum principle

Let M be a Hilbert manifold and g_t $(0\leq t<T)$ a C^∞-family of Riemannian metrics on M and $G \curvearrowright M$ a almost free action which is isometric with respect to g_t's $(t\in[0,T))$. Assume that the orbit space M/G is compact. Let $\mathcal{H}_t$ $(0\leq t<T)$ be the horizontal distribution of the G-action and define a subbundle $\mathcal{H}$ of π_M^*TM by $\mathcal{H}_{(x,t)}:=(\mathcal{H}_t)_x$. For a tangent vector field X on M (or an open set U of M), we define a section $\bar{X}$ of π_M^*TM (or $\pi_M^*TM|_U$) by $\bar{X}_{(x,t)}:=X_x$ $((x,t)\in M\times[0,T))$. Let ∇^t $(0\leq t<T)$ be the Riemannian connection of g_t and ∇ the connection of π_M^*TM defined in terms of ∇^t's $(t\in[0,T))$. Define a connection $\nabla^{\mathcal{H}}$ of $\mathcal{H}$ by $\nabla^{\mathcal{H}}_XY=\mathrm{pr}_{\mathcal{H}}(\nabla_XY)$ for any $X\in T(M\times[0,T))$ and any $Y\in\Gamma(\mathcal{H})$. For $B\in\Gamma(\pi_M^*T^{(r_0,s_0)}M)$, we define maps $\psi_{B\otimes}$ and $\psi_{\otimes B}$ from $\Gamma(\pi_M^*T^{(r,s)}M)$ to $\Gamma(\pi_M^*T^{(r+r_0,s+s_0)}M)$ by

$$\psi_{B\otimes}(S):=B\otimes S,\quad\text{and}\quad\psi_{\otimes B}(S):=S\otimes B\quad(S\in\Gamma(\pi_M^*T^{(r,s)}M),$$

respectively. Also, we define a map $\psi_{\otimes^k}$ of $\Gamma(\pi_M^*T^{(r,s)}M)$ to $\Gamma(\pi_M^*T^{(kr,ks)}M)$ by

$$\psi_{\otimes^k}(S):=S\otimes\cdots\otimes S\ (k-\text{times})\quad(S\in\Gamma(\pi_M^*T^{(r,s)}M).$$

Also, we define a map $\psi_{g_{\mathcal{H}},ij}$ $(i<j)$ from $\Gamma(\pi_M^*T^{(0,s)}M)$ (or $\Gamma(\pi_M^*T^{(1,s)}M)$) to $\Gamma(\pi_M^*T^{(0,s-2)}M)$ (or $\Gamma(\pi_M^*T^{(1,s-2)}M)$) by

$$\begin{aligned}&(\psi_{g_{\mathcal{H}},ij}(S))_{(x,t)}(X_1,\cdots,X_{s-2})\\ :=&\sum_{k=1}^n S_{(x,t)}(X_1,\cdots,X_{i-1},e_k,X_{i+1},\cdots,X_{j-1},e_k,X_{j+1},\cdots,X_{s-2})\end{aligned}$$

and define a map $\psi_{\mathcal{H},i}$ from $\Gamma(\pi_M^*T^{(1,s)}M)$ to $\Gamma(\pi_M^*T^{(0,s-1)}M)$ by

$$(\psi_{\mathcal{H},i}(S))_{(x,t)}(X_1,\cdots,X_{s-1}):=\mathrm{Tr}\,S_{(x,t)}(X_1,\cdots,X_{i-1},\bullet,X_i,\cdots,X_{s-1}),$$

where $X_i\in T_xM$ $(i=1,\cdots,s-1)$ and $\{e_1,\cdots,e_n\}$ is an orthonormal base of $(\mathcal{H}_t)_x$ with respect to g_t. We call a map P from $\Gamma(\pi_M^*T^{(0,s)}M)$ to oneself given by the composition of the above maps of five type *a map of polynomial type*.

In [K], we proved the following maximum principle for a C^∞-family of G-invariant $(0,2)$-tensor fields on M.

Theorem 4.1. *Let $S \in \Gamma(\pi_M^*(T^{(0,2)}M))$ such that, for each $t \in [0,T)$, $S_t(:= S_{(\cdot,t)})$ is a G-invariant symmetric $(0,2)$-tensor field on M. Assume that S_t's $(0 \le t < T)$ satisfy the following evolution equation:*

$$\frac{\partial S_{\mathcal{H}}}{\partial t} = \triangle_{\mathcal{H}}^{\mathcal{H}} S_{\mathcal{H}} + \nabla_{\bar{X}_0}^{\mathcal{H}} S_{\mathcal{H}} + P(S)_{\mathcal{H}}, \tag{5.1}$$

where $X_0 \in \Gamma(TM)$ and P is a map of polynomial type from $\Gamma(\pi_M^*(T^{(0,2)}M))$ to oneself.

(i) *Assume that P satisfies the following condition:*

$$X \in \mathrm{Ker}(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)} \ \Rightarrow \ P(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)}(X,X) \ge 0$$

for any $\varepsilon > 0$ and any $(x,t) \in M \times [0,T)$. Then, if $(S_{\mathcal{H}})_{(\cdot,0)} \ge 0$ (resp. > 0), then $(S_{\mathcal{H}})_{(\cdot,t)} \ge 0$ (resp. > 0) holds for all $t \in [0,T)$.

(ii) *Assume that P satisfies the following condition:*

$$X \in \mathrm{Ker}(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)} \ \Rightarrow \ P(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)}(X,X) \le 0$$

for any $\varepsilon > 0$ and any $(x,t) \in M \times [0,T)$. Then, if $(S_{\mathcal{H}})_{(\cdot,0)} \le 0$ (resp. < 0), then $(S_{\mathcal{H}})_{(\cdot,t)} \le 0$ (resp. < 0) holds for all $t \in [0,T)$.

Similarly we obtain the following maximal principle for a C^∞-family of G-invariant functions on M.

Theorem 4.2. *Let ρ be a C^∞-function over $M \times [0,T)$ such that, for each $t \in [0,T)$, $\rho_t(:= \rho(\cdot,t))$ is a G-invariant function on M. Assume that ρ_t's $(0 \le t < T)$ satisfy the following evolution equation:*

$$\frac{\partial \rho}{\partial t} = \triangle_{\mathcal{H}} \rho + d\rho(\bar{X}_0) + P(\rho),$$

where $X_0 \in \Gamma(TM)$ and P is a map of polynomial type from $C^\infty(M \times [0,T))$ to oneself.

(i) *Assume that P satisfies the following condition:*

$$(\rho + \varepsilon)_{(x,t)} = 0 \ \Rightarrow \ P(\rho + \varepsilon)_{(x,t)} \ge 0$$

for any $\varepsilon > 0$ and any $(x,t) \in M \times [0,T)$. Then, if $\rho_0 \ge 0$ (resp. > 0), then $\rho_t \ge 0$ (resp. > 0) holds for all $t \in [0,T)$.

(ii) *Assume that P satisfies the following condition:*

$$(\rho + \varepsilon)_{(x,t)} = 0 \ \Rightarrow \ P(\rho + \varepsilon)_{(x,t)} \le 0$$

for any $\varepsilon > 0$ and any $(x,t) \in M \times [0,T)$. Then, if $\rho_0 \le 0$ (resp. < 0), then $\rho_t \le 0$ (resp. < 0) holds for all $t \in [0,T)$.

5 Horizontally strongly convexity preservability theorem

Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$ and $\phi : V \to N := V/G$ the orbit map. Denote by $\widetilde{\nabla}$ the Riemannian connection of V. Set $n := \dim N - 1$. Let $M(\subset V)$ be a G-invariant hypersurface in V such that $\phi(M)$ is compact. Let f be an inclusion map of M into V and f_t $(0 \leq t < T)$ the regularized mean curvature flow starting from f. We use the notations in Sections 3. In the sequel, we omit the notation f_{t*} for simplicity. Set

$$L := \sup_{u \in V} \max_{(X_1,\cdots,X_5) \in (\widetilde{\mathcal{H}}_1)_u^5} |\langle \mathcal{A}^{\phi}_{X_1}((\widetilde{\nabla}_{X_2}\mathcal{A}^{\phi})_{X_3}X_4),\, X_5\rangle|,$$

where $\widetilde{\mathcal{H}}_1 := \{X \in \widetilde{\mathcal{H}} \mid\ ||X|| = 1\}$. Assume that $L < \infty$. Note that $L < \infty$ in the case where N is compact. In [K], we proved the following horizontally strongly convexity preservability theorem by using evolution equations stated in Section 3 and the discussion in the proof of Theorem 4.1.

Theorem 5.1. *If M satisfies $||H_0||^2(h_{\mathcal{H}})_{(\cdot,0)} > 2n^2L(g_{\mathcal{H}})_{(\cdot,0)}$, then $T < \infty$ holds and $||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)} > 2n^2L(g_{\mathcal{H}})_{(\cdot,t)}$ holds for all $t \in [0,T)$.*

6 A collapsing theorem

Let $G \curvearrowright V$, ϕ, M, f, f_t, ϕ_M $(0 \leq t < T)$ and L be as in the previous section. In this section, we use the notations in Sections 3 and 5. Let $\overline{K}$ be the maximal sectional curvature of (N, g_N) and $\overline{R}(\bullet)$ the injective radius of (N, g_N) restricted to $\bullet(\subset N)$. Set $b := \sqrt{\overline{K}}(\in \mathbb{R} \cup \sqrt{-1}\mathbb{R})$. For f and $0 < \alpha < 1$, we consider the following conditions:

$$(*_1^{\alpha}) \qquad b^2(1-\alpha)^{-2/n}(\omega_n^{-1} \cdot \mathrm{Vol}_{\overline{g}}(\overline{M}))^{2/n} \leq 1$$

and

$$(*_2^{\alpha}), \qquad b^{-1}\sin^{-1} b \cdot (1-\alpha)^{-1/n} \cdot (\omega_n^{-1} \cdot \mathrm{Vol}_{\overline{g}}(\overline{M}))^{1/n} < \frac{1}{2}\overline{R}((\phi(f(M))),$$

where $\overline{g}$ is the induced metric on $\overline{M} = \phi_M(M)$ by $\bar{f}$ and ω_n is the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Set $||H_t||_{\min} := \min_M ||H_t||$ and $||H_t||_{\max} := \max_M ||H_t||$.

The main purpose of this paper is to prove the following collapsing theorem.

Theorem 6.1. *Assume that $f(=f_0)$ satisfies the above conditions $(*_1^\alpha), (*_2^\alpha)$ and $||H_0||^2(h_{\mathcal{H}})_{(\cdot,0)} > 2n^2 L(g_{\mathcal{H}})_{(\cdot,0)}$. Then the following statements* (i) *and* (ii) *hold:*
(i) *$T < \infty$ and $f_t(M)$ collapses to some G-orbit as $t \to T$.*
(ii) $\displaystyle\lim_{t\to T} \frac{||H_t||_{\max}}{||H_t||_{\min}} = 1$ *holds.*

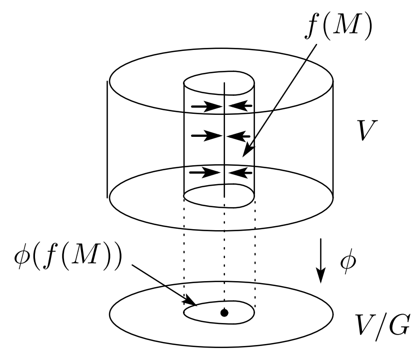
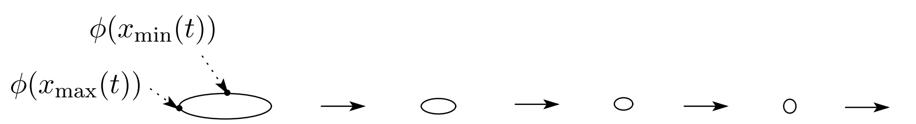


Figure 1.

Remark 6.1. "$\displaystyle\lim_{t\to T} \frac{||H_t||_{\max}}{||H_t||_{\min}} = 1$" implies that $f_t(M)$ converges to an infinitesimal constant tube over some G-orbit as $t \to T$ (or equivalently, $\phi(f_t(M))$ converges to a round point(=an infinitesimal round sphere) as $t \to T$) (see Figure 2).



$x_{\max}(t)$: a point attaining $||H_t||_{\max}$
$x_{\min}(t)$: a point attaining $||H_t||_{\min}$

Figure 2.

In Sections 7-8, we shall derive some important facts to prove the statement (ii) of this theorem.

7 Approach to horizontally totally umbilicity

Let f and f_t $(0 \leq t < T)$ be as in the statement of Thoerem 6.1. Then, according to Theorem 5.1, for all $t \in [0,T)$,

$$||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)} > 2n^2 L(g_{\mathcal{H}})_{(\cdot,t)} \tag{7.1}$$

holds.

In this section, we shall prove the following result for the approach to the horizontally totally umbilicity of f_t as $t \to T$.

Proposition 7.1. *Under the hypothesis of Theorem 6.1, there exist positive constants δ and C_0 depending on only f, L, K and the injective radius $i(N)$ of N such that*

$$||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n} < C_0||H||^{2-\delta}$$

holds for all $t \in [0,T)$.

We prepare some lemmas to show this proposition. In the sequel, we denote the fibre metric of $\mathcal{H}^{(r,s)}$ induced from $g_{\mathcal{H}}$ by the same symbol $g_{\mathcal{H}}$, and set $||S|| := \sqrt{g_{\mathcal{H}}(S,S)}$ for $S \in \Gamma(\mathcal{H}^{(r,s)})$. Define a function ψ_δ over M by

$$\psi_\delta := \frac{1}{||H||^{2-\delta}}\left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n}\right).$$

Lemma 7.1.1. *Set $\alpha := 2-\delta$. Then we have*

$$\begin{aligned}
\frac{\partial \psi_\delta}{\partial t} = {} & \triangle_{\mathcal{H}}\psi_\delta + (2-\alpha)||A_{\mathcal{H}}||^2\psi_\delta + \frac{(\alpha-1)(\alpha-2)}{||H||^2}||\, d||H||\, ||^2\psi_\delta \\
& + \frac{2(\alpha-1)}{||H||} g_{\mathcal{H}}(d||H||, d\psi_\delta) - \frac{2}{||H||^{\alpha+2}}\left|\left|\, ||H||\nabla^{\mathcal{H}}A_{\mathcal{H}} - d||H|| \otimes A_{\mathcal{H}}\right|\right|^2 \\
& + 3(\alpha-2)\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal{H}})\psi_\delta - \frac{6}{||H||^{\alpha-1}}\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2 \circ A_{\mathcal{H}}) \\
& + \frac{6}{n||H||^{\alpha-2}}\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2) - \frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h(((\nabla_\bullet \mathcal{A})_\bullet \circ A_{\mathcal{H}})\cdot, \cdot) \\
& - \frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\mathcal{A}_\bullet \circ A_{\mathcal{H}})\cdot, \mathcal{A}_\bullet \cdot) + \frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}} h((\mathcal{A}_\bullet \circ \mathcal{A}_\bullet)\cdot, A_{\mathcal{H}}\cdot).
\end{aligned}$$

Proof. By using Lemmas 3.8 and 3.10, we have

$$
\begin{aligned}
\frac{\partial\psi_\delta}{\partial t} =& (2-\alpha)||A_{\mathcal H}||^2\psi_\delta + \frac{1}{||H||^\alpha}\triangle_{\mathcal H}(||A_{\mathcal H}||^2)\\
&-\frac{1}{||H||^{\alpha+1}}\left(\alpha||A_{\mathcal H}||^2 - \frac{(\alpha-2)||H||^2}{n}\right)\triangle_{\mathcal H}||H||\\
&-\frac{2}{||H||^\alpha}||\nabla^{\mathcal H}A_{\mathcal H}||^2 + (3\alpha-2)\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\cdot\psi_\delta\\
&-\frac{6}{||H||^{\alpha-1}}\mathrm{Tr}\left((\mathcal{A}^\phi_\xi)^2\circ(A_{\mathcal H} - \frac{||H||}{n}\,\mathrm{id})\right)\\
&-\frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal H}}\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}h((\mathcal{A}_\bullet\circ A_{\mathcal H})\cdot,\mathcal{A}_\bullet\cdot)\\
&+\frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal H}}\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}h((\mathcal{A}_\bullet\circ \mathcal{A}_\bullet)\cdot,A_{\mathcal H}\cdot)\\
&-\frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal H}}\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}h((\nabla_\bullet\mathcal{A})_\bullet\circ A_{\mathcal H})\cdot,\cdot).
\end{aligned}
\tag{7.2}
$$

Also we have

$$
\begin{aligned}
\triangle_{\mathcal H}\psi_\delta =& \frac{1}{||H||^\alpha}\triangle_{\mathcal H}||A_{\mathcal H}||^2 - \frac{2\alpha}{||H||^{\alpha+1}}g_{\mathcal H}(d||H||, d(||A_{\mathcal H}||^2))\\
&-\frac{1}{||H||^{\alpha+1}}\left(\alpha||A_{\mathcal H}||^2 - \frac{(\alpha-2)||H||^2}{n}\right)\triangle_{\mathcal H}||H||\\
&+\frac{1}{||H||^{\alpha+2}}\left(\alpha(\alpha+1)||A_{\mathcal H}||^2 - \frac{(\alpha-1)(\alpha-2)||H||^2}{n}\right)||d||H||\,||^2
\end{aligned}
\tag{7.3}
$$

From (7.2) and (7.3), we obtain the desired relation. q.e.d.

Then we have the following inequalities.

By using the Codazzi equation, we can derive the following relation.

Lemma 7.1.2. *For any $X, Y, Z \in \mathcal{H}$, we have*

$$(\nabla^{\mathcal H}_X h_{\mathcal H})(Y,Z) = (\nabla^{\mathcal H}_Y h_{\mathcal H})(X,Z) + 2h(\mathcal{A}_XY,Z) - h(\mathcal{A}_YZ,X) + h(\mathcal{A}_XZ,Y)$$

or equivalently,

$$(\nabla^{\mathcal H}_X A_{\mathcal H})(Y) = (\nabla^{\mathcal H}_Y A_{\mathcal H})(X) + 2(A\circ\mathcal{A}_X)Y + (\mathcal{A}_Y\circ A)(X) - (\mathcal{A}_X\circ A)(Y).$$

Proof. Let (x,t) be the base point of X, Y and Z and extend these vectors to sections $\widetilde{X}, \widetilde{Y}$ and $\widetilde{Z}$ of $\mathcal{H}_t$ with $(\nabla^{\mathcal H}\widetilde{X})_{(x,t)} = (\nabla^{\mathcal H}\widetilde{Y})_{(x,t)} = (\nabla^{\mathcal H}\widetilde{Z})_{(x,t)} = 0$. Since ∇h is

symmetric with respect to g by the Codazzi equation and the flatness of V, we have

$$\begin{aligned}
&(\nabla^{\mathcal{H}}_X h_{\mathcal{H}})(Y,Z) = X(h(\widetilde{Y},\widetilde{Z}))\\
&= (\nabla_X h)(Y,Z) + h(\mathcal{A}_X Y, Z) + h(\mathcal{A}_X Z, Y)\\
&= (\nabla_Y h)(X,Z) + h(\mathcal{A}_X Y, Z) + h(\mathcal{A}_X Z, Y)\\
&= Y(h(\widetilde{X},\widetilde{Z})) - h(\mathcal{A}_Y X, Z) - h(\mathcal{A}_Y Z, X) + h(\mathcal{A}_X Y, Z) + h(\mathcal{A}_X Z, Y)\\
&= (\nabla^{\mathcal{H}}_Y h)(X,Z) + 2h(\mathcal{A}_X Y, Z) - h(\mathcal{A}_Y Z, X) + h(\mathcal{A}_X Z, Y).
\end{aligned}$$

q.e.d.

Set

$$K := \max_{(e_1,e_2):\text{o.n.s. of } TV} ||\mathcal{A}^{\phi}_{e_1} e_2||^2,$$

where "o.n.s." means "orthonormal system". Assume that $K < \infty$. Note that $K < \infty$ if $N = V/G$ is compact. For a section S of $\mathcal{H}^{(r,s)}$ and a permutaion σ of s-symbols, we define a section S_σ of $\mathcal{H}^{(r,s)}$ by

$$S_\sigma(X_1,\cdots,X_s) := S(X_{\sigma(1)},\cdots,X_{\sigma(s)}) \quad (X_1,\cdots,X_s \in \mathcal{H})$$

and $\mathrm{Alt}(S)$ by

$$\mathrm{Alt}(S) := \frac{1}{s!}\sum_\sigma \mathrm{sgn}\,\sigma\, S_\sigma,$$

where σ runs over the symmetric group of degree s. Also, denote by (i,j) the transposition exchanging i and j. Since $||H_t||^2(h_{\mathcal{H}})_{(\cdot,t)} > n^2 L(g_{\mathcal{H}})_{(\cdot,t)}$ $(t \in [0,T))$ and $\phi(M)$ is compact, for each $t \in [0,T)$, there exists a positive constant ε_t satisfying

$$(\sharp) \qquad ||H_{(\cdot,t)}||^2(h_{\mathcal{H}})_{(\cdot,t)} \geq n^2 L(g_{\mathcal{H}})_{(\cdot,t)} + \varepsilon_t ||H_{(\cdot,t)}||^3 (g_{\mathcal{H}})_{(\cdot,t)}.$$

Define a function ε over $[0,T)$ by $\varepsilon(t) := \varepsilon_t$ $(t \in [0,T))$. Without loss of generality, we may assume that ε is continuous and $\varepsilon \leq 1$. Then we have the following inequalities.

Lemma 7.1.3. *Let ε be as above. Then we have the following inequalities:*

$$(7.4) \qquad ||H||\mathrm{Tr}_{\mathcal{H}}(A_{\mathcal{H}})^3 - ||(A_{\mathcal{H}})_t||^4 \geq n\varepsilon^2 ||H||^2 \left(||(A_{\mathcal{H}})_t||^2 - \frac{||H||^2}{n}\right),$$

and

$$(7.5) \qquad \begin{aligned}&\Big|\Big|\ ||H||\nabla^{\mathcal{H}} A_{\mathcal{H}} - d||H|| \otimes A_{\mathcal{H}}\ \Big|\Big|^2\\ &\geq -8\varepsilon^{-2} K ||H_{(u,t)}||^2 + \frac{1}{8}||(d||H||)_{(u,t)}||^2 \varepsilon^2 ||H_{(u,t)}||^2.\end{aligned}$$

Proof. First we shall show the inequality (7.4). Fix $(u,t) \in M \times [0,T)$. Take an orthonormal base $\{e_1, \cdots, e_n\}$ of $\mathcal{H}_{(u,t)}$ with respect to $g_{(u,t)}$ consisting of the eigenvectors of $(A_{\mathcal{H}})_{(u,t)}$. Let $(A_{\mathcal{H}})_{(u,t)}(e_i) = \lambda_i e_i$ $(i = 1, \cdots, n)$. Note that $\lambda_i > \varepsilon||H||(> 0)$ $(i = 1, \cdots, n)$. Then we have

$$||H||\mathrm{Tr}_{\mathcal{H}}(A_{\mathcal{H}})^3 - ||(A_{\mathcal{H}})_t||^4 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 > \varepsilon^2 ||H||^2 \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2.$$

On the other hand, we have

$$||(A_{\mathcal{H}})_t||^2 - \frac{||H||^2}{n} = \frac{1}{n} \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2.$$

From these inequalities, we can derive the inequality (7.4).

Next we shall show the inequality (7.5). By using Lemma 7.1.2, we can show

$$\begin{aligned}
&\big|\big|\ ||H||\nabla^{\mathcal{H}} A_{\mathcal{H}} - d||H|| \otimes A_{\mathcal{H}}\ \big|\big|^2 \geq \big|\big|\mathrm{Alt}\left(||H||\nabla^{\mathcal{H}} A_{\mathcal{H}} - d||H|| \otimes A_{\mathcal{H}}\right)\big|\big|^2 \\
&\geq \left|\left| A \circ \mathcal{A} - \frac{1}{2}\mathcal{A} \circ (A \times \mathrm{id}) - \frac{1}{2}\mathcal{A} \circ (\mathrm{id} \times A) - \mathrm{Alt}\left(d||H|| \otimes A_{\mathcal{H}}\right)\right|\right|^2.
\end{aligned}$$

For simplicity, we set

$$S := A \circ \mathcal{A} - \frac{1}{2}\mathcal{A} \circ (A \times \mathrm{id}) - \frac{1}{2}\mathcal{A} \circ (\mathrm{id} \times A).$$

It is clear that (7.5) holds at (u,t) if $(d||H||)_{(u,t)} = 0$. Assume that $(d||H||)_{(u,t)} \neq 0$. Take an orthonormal base $(e_1, \cdots, e_n)$ of $\mathcal{H}_{(u,t)}$ with respect to $(g_{\mathcal{H}})_{(u,t)}$ with $e_1 = \frac{(d||H||)_{(u,t)}}{||(d||H||)_{(u,t)}||}$. Then we have

$$\begin{aligned}
&\left|\left| S_{(u,t)} - \mathrm{Alt}\left(d||H|| \otimes A_{\mathcal{H}}\right)_{(u,t)} \right|\right|^2 \\
&\geq ||S - \mathrm{Alt}\left(d||H|| \otimes A_{\mathcal{H}}\right)(e_1, e_2)||^2 \\
&\geq ||S(e_1, e_2)||^2 - ||(d||H||)_{(u,t)}|| g(S(e_1, e_2), A_{\mathcal{H}} e_2) + \frac{1}{4}||(d||H||)_{(u,t)}||^2 \cdot ||A_{\mathcal{H}} e_2||^2 \\
&\geq ||S(e_1, e_2)||^2 - ||(d||H||)_{(u,t)}|| g(S(e_1, e_2), A_{\mathcal{H}} e_2) + \frac{1}{4}||(d||H||)_{(u,t)}||^2 \varepsilon^2 ||H_{(u,t)}||^2 \\
&\geq (1 - 2\varepsilon^{-2})||S(e_1, e_2)||^2 + \left(\sqrt{2}\varepsilon^{-1}||S(e_1, e_2)|| - \frac{1}{2\sqrt{2}}||(d||H||)_{(u,t)}||\varepsilon||H_{(u,t)}||\right)^2 \\
&\quad + \frac{1}{8}||(d||H||)_{(u,t)}||^2 \varepsilon^2 ||H_{(u,t)}||^2 \\
&\geq -2\varepsilon^{-2}||S(e_1, e_2)||^2 + \frac{1}{8}||(d||H||)_{(u,t)}||^2 \varepsilon^2 ||H_{(u,t)}||^2 \\
&\geq -8\varepsilon^{-2} K ||H_{(u,t)}||^2 + \frac{1}{8}||(d||H||)_{(u,t)}||^2 \varepsilon^2 ||H_{(u,t)}||^2,
\end{aligned}$$

where we use $||A_{\mathcal{H}}e|| \leq ||H||$ holds for any unit vector e of $\mathcal{H}$. Thus we see that (7.5) holds at (u,t). This completes the proof. q.e.d.

From Lemma 7.1.1 and (7.5), we obtain the following lemma.

Lemma 7.1.4. *Assume that $\delta < 1$. Then we have the following inequality:*

$$\begin{aligned}
\frac{\partial\psi_\delta}{\partial t} \leq\; & \triangle_{\mathcal{H}}\psi_\delta + (2-\alpha)||A_{\mathcal{H}}||^2\psi_\delta + \frac{2(\alpha-1)}{||H||}g_{\mathcal{H}}(d||H||, d\psi_\delta)\\
& -\frac{2}{||H||^{\alpha+2}}\left(\frac{1}{8}||\,d||H||\,||^2\varepsilon^2||H||^2 - 8\varepsilon^{-2}K||H||^2\right)\\
& +3(\alpha-2)\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal{H}})\psi_\delta - \frac{6}{||H||^{\alpha-2}}\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2\circ A_{\mathcal{H}})\\
& +\frac{6}{n||H||^{\alpha-2}}\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2) - \frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(((\nabla_\bullet\mathcal{A})_\bullet\circ A_{\mathcal{H}})\cdot,\cdot)\\
& -\frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ A_{\mathcal{H}})\cdot,\mathcal{A}_\bullet\cdot)\\
& +\frac{4}{||H||^\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ\mathcal{A}_\bullet)\cdot, A_{\mathcal{H}}\cdot).
\end{aligned}$$

On the other hand, we can show the following fact for ψ_δ.

Lemma 7.1.5. *We have*

$$\begin{aligned}
\triangle_{\mathcal{H}}\psi_\delta =\; & \frac{2}{||H||^{\alpha+2}}\times\left|\left|\; ||H||\cdot\nabla^{\mathcal{H}}A_{\mathcal{H}} - d||H||\cdot A_{\mathcal{H}}\;\right|\right|^2\\
& +\frac{2}{||H||^{\alpha-1}}\left(\mathrm{Tr}((A_{\mathcal{H}})^3) - \mathrm{Tr}((\mathcal{A}^\phi_\xi)^2\circ A_{\mathcal{H}})\right)\\
& -\frac{2}{||H||^\alpha}\left(\mathrm{Tr}((A_{\mathcal{H}})^2 - (\mathcal{A}^\phi_\xi)^2|_{\mathcal{H}}\right)||A_{\mathcal{H}}||^2\\
& +\frac{2}{||H||^\alpha}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\left((\nabla^{\mathcal{H}}d||H||)\left(\left(A_{\mathcal{H}} - \frac{||H||}{n}\,\mathrm{id}\right)(\bullet),\bullet\right)\right)\\
& -\frac{\alpha}{||H||}\psi_\delta\triangle_{\mathcal{H}}||H|| - \frac{(\alpha-1)(\alpha-2)}{||H||^2}||\,d||H||\,||^2\psi_\delta\\
& -\frac{2(\alpha-1)}{||H||}g_{\mathcal{H}}(d||H||, d\psi_\delta) + \frac{2}{||H||^\alpha}\times\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet,\bullet).
\end{aligned}$$

Proof. According to (4.16) in [K-MFHI], we have

$$\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}(\triangle^{\mathcal{H}}_{\mathcal{H}}h_{\mathcal{H}})(A_{\mathcal{H}}\bullet,\bullet) = \frac{1}{2}\triangle_{\mathcal{H}}||A_{\mathcal{H}}||^2 - ||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2. \tag{7.6}$$

Also we have

$$(A^2)_{\mathcal{H}} = (A_{\mathcal{H}})^2 - (\mathcal{A}^\phi_\xi)^2. \tag{7.7}$$

By using Lemmas 3.3, 3.4 and these relations, we can derive

$$\begin{aligned}
&\frac{1}{2}\triangle_{\mathcal{H}}||A_{\mathcal{H}}||^2 = \mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\nabla^{\mathcal{H}} d||H||)(A_{\mathcal{H}}\bullet, \bullet) + ||H||\mathrm{Tr}((A_{\mathcal{H}})^3)\\
&-||H||\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2 \circ A_{\mathcal{H}}) - \mathrm{Tr}\left((A_{\mathcal{H}})^2 - (\mathcal{A}^\phi_\xi)^2|_{\mathcal{H}}\right)||A_{\mathcal{H}}||^2\\
&+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet, \bullet) + ||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2.
\end{aligned} \tag{7.8}$$

By substituting this relation into (7.3), we obtain

$$\begin{aligned}
\triangle_{\mathcal{H}}\psi_\delta = \frac{2}{||H||^\alpha} \times \{&\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}(\nabla^{\mathcal{H}} d||H||)(A_{\mathcal{H}}\bullet, \bullet) + ||H||\mathrm{Tr}((A_{\mathcal{H}})^3)\\
&-||H||\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2 \circ A_{\mathcal{H}}) - \mathrm{Tr}\left((A_{\mathcal{H}})^2 - (\mathcal{A}^\phi_\xi)^2|_{\mathcal{H}}\right)||A_{\mathcal{H}}||^2\\
&+\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet, \bullet) + ||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2\}\\
-\frac{2\alpha}{||H||^{\alpha+1}}&g_{\mathcal{H}}(d||H||, d(||A_{\mathcal{H}}||^2))\\
-\frac{1}{||H||^{\alpha+1}}&\left(\alpha||A_{\mathcal{H}}||^2 - \frac{(\alpha-2)||H||^2}{n}\right)\triangle_{\mathcal{H}}||H||\\
+\frac{1}{||H||^{\alpha+2}}&\left(\alpha(\alpha+1)||A_{\mathcal{H}}||^2 - \frac{(\alpha-1)(\alpha-2)||H||^2}{n}\right)||d||H||\,||^2.
\end{aligned}$$

From this relation, we can derive the desired relation. q.e.d.

From this lemma, we can derive the following inequality for ψ_δ directly.

Lemma 7.1.6. *We have*

$$\begin{aligned}
\triangle_{\mathcal{H}}\psi_\delta \geq &\frac{2}{||H||^{\alpha-1}}\left(\mathrm{Tr}((A_{\mathcal{H}})^3) - \mathrm{Tr}((\mathcal{A}^\phi_\xi)^2 \circ A_{\mathcal{H}})\right)\\
&-\frac{2}{||H||^\alpha}\left(||A_{\mathcal{H}}||^2 - \mathrm{Tr}(\mathcal{A}^\phi_\xi)^2|_{\mathcal{H}}\right)\mathrm{Tr}((A_{\mathcal{H}})^2)\\
&+\frac{2}{||H||^\alpha}\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\left((\nabla^{\mathcal{H}} d||H||)\left(\left(A_{\mathcal{H}} - \frac{||H||}{n}\,\mathrm{id}\right)(\bullet), \bullet\right)\right)\\
&-\frac{\alpha}{||H||}\psi_\delta\triangle_{\mathcal{H}}||H|| - \frac{2(\alpha-1)}{||H||}g_{\mathcal{H}}(d||H||, d\psi_\delta)\\
&+\frac{2}{||H||^\alpha} \times \mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet, \bullet).
\end{aligned}$$

For a function ρ over $M \times [0,T)$ such that $\rho(\cdot,t)$ $(t \in [0,T))$ are G-invariant, define a function ρ_B over $\overline{M} \times [0,T)$ by $\rho_B \circ (\phi_M \times \mathrm{id}_{[0,T)}) = \rho$. We call this function the *function over* $\overline{M} \times [0,T)$ *associated with* ρ. Denote by g_N the Riemannian orbimetric of N and set $\bar{g}_t := \bar{f}_t^* g_N$. Also, denote by $d\bar{v}_t$ the orbivolume element of $\bar{g}_t$. Define a section $\bar{g}$ of $\pi_{\overline{M}}^*(T^{(0,2)}\overline{M})$ by $\bar{g}(x,t) = (g_t)_x$ $((x,t) \in \overline{M} \times [0,T))$ and a section $d\bar{v}$ of $\pi_{\overline{M}}^*(\wedge^n T^*\overline{M})$ by $d\bar{v}(x,t) = (d\bar{v}_t)_x$ $((x,t) \in \overline{M} \times [0,T))$, where $\pi_{\overline{M}}$ is the natural projection of $\overline{M} \times [0,T)$ onto $\overline{M}$ and $\pi_{\overline{M}}^*(\bullet)$ denotes the induced bundle of $(\bullet)$ by $\pi_{\overline{M}}$. Denote by $\overline{\nabla}^t$ the Riemannian orbiconnection of $\bar{g}_t$ and by $\overline{\triangle}_t$ the Laplace operator of $\overline{\nabla}^t$. Define an orbiconnection $\overline{\nabla}$ of $\pi_{\overline{M}}^*(T\overline{M})$ by using $\overline{\nabla}^t$'s (see the definition of ∇ in Section 3). Also, let $\overline{\triangle}$ be the differential operator of $\pi_{\overline{M}}^*(\overline{M} \times \mathbb{R})$ defined by using $\overline{\triangle}_t$'s. Denote by $\int_{\overline{M}} \rho_B \, d\bar{v}$ the function over $[0,T)$ defined by assigning $\int_{\overline{M}} \rho_B(\cdot,t)\, d\bar{v}_t$ to each $t \in [0,T)$. Clearly we have

$$\int_{\overline{M}} (\mathrm{div}_{\nabla^{\mathcal{H}}} \rho)_B d\bar{v} = \int_{\overline{M}} \mathrm{div}_{\overline{\nabla}}(\rho_B) d\bar{v} = 0 \tag{7.9}$$

and

$$\int_{\overline{M}} (\triangle_{\mathcal{H}} \rho)_B d\bar{v} = \int_{\overline{M}} \overline{\triangle}(\rho_B) d\bar{v} = 0. \tag{7.10}$$

From the inequlaity in Lemma 7.1.6 and (7.9), we can derive the following integral inequality.

Lemma 7.1.7. *Assume that $0 \leq \delta \leq \frac{1}{2}$. Then, for any $\beta \geq 2$, we have*

$$\begin{aligned}
&n\varepsilon^2 \int_{\overline{M}} ||H||_B^2 (\psi_\delta)_B^\beta d\bar{v} \\
\leq\ & \frac{3\beta\eta + 6}{2} \int_{\overline{M}} ||H||_B^{-\alpha} (\psi_\delta)_B^{\beta-1} ||\ d||H||\ ||_B^2 d\bar{v} + \frac{3\beta}{2\eta} \int_{\overline{M}} (\psi_\delta)_B^{\beta-2} ||d\psi_\delta||_B^2 d\bar{v} \\
&+ C_1 \int_{\overline{M}} ||H||_B^{-\alpha} (\psi_\delta)_B^{\beta-1} ||A_{\mathcal{H}}||_B^2 d\bar{v} + C_2 \int_{\overline{M}} ||H||_B^{-\alpha} (\psi_\delta)_B^{\beta-1} d\bar{v},
\end{aligned}$$

where C_i $(i = 1,2)$ are positive constants depending only on K and L (L is the constant defined in the previous section).

Proof. By using $\int_{\overline{M}} \mathrm{div}_{\nabla^{\mathcal{H}}} \left(||H||^{-\alpha} (\psi_\delta)^{\beta-1} (A_{\mathcal{H}} - (||H||/n)\mathrm{id})(\mathrm{grad}\,||H||) \right)_B d\bar{v} = 0$

and Lemma 7.1.2, we can show

$$
\begin{aligned}
&\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}\left(\mathrm{Tr}^{\bullet}_{g_{\mathcal H}}\left((\nabla^{\mathcal H}d||H||)\left((A_{\mathcal H}-(||H||/n)\,\mathrm{id})\,(\bullet),\bullet\right)\right)\right)_B d\bar v\\
&=\alpha\int_{\overline{M}}||H||_B^{-\alpha-1}(\psi_\delta)_B^{\beta-1}g_{\mathcal H}((d||H||\otimes d||H||,h_{\mathcal H}-(||H||/n)g_{\mathcal H})_B d\bar v\\
&\quad-(\beta-1)\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-2}g_{\mathcal H}((d||H||\otimes d\psi_\delta,h_{\mathcal H}-(||H||/n)g_{\mathcal H})_B d\bar v\\
&\quad-(1-1/n)\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||\,d||H||\;||_B^2 d\bar v\\
&\quad+3\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}\mathrm{Tr}_{g_{\mathcal H}}(\mathcal{A}^{\phi}_{\xi}\circ\mathcal{A}^{\phi}_{\mathrm{grad}||H||})_B d\bar v.
\end{aligned}
\tag{7.11}
$$

Also, by using $\int_{\overline{M}}(\triangle_{\mathcal H}\psi_\delta^{\beta})_B d\bar v=0$, we can show

$$
\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}(\triangle_{\mathcal H}\psi_\delta)_B d\bar v=-(\beta-1)\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar v
\tag{7.12}
$$

and hence

$$
\begin{aligned}
&\int_{\overline{M}}||H||_B^{-1}(\psi_\delta)_B^{\beta}(\triangle_{\mathcal H}||H||)_B d\bar v\\
&=-2\beta\int_{\overline{M}}||H||_B^{-1}(\psi_\delta)_B^{\beta-1}g_{\mathcal H}(d||H||,d\psi_\delta)_B d\bar v\\
&\quad+2\int_{\overline{M}}||H||_B^{-2}(\psi_\delta)_B^{\beta}||\,d||H||\;||_B^2 d\bar v.
\end{aligned}
\tag{7.13}
$$

By multiplying $\psi_\delta^{\beta-1}$ to both sides of the inequality in Lemma 7.1.6 and integrating the functions over $\overline{M}$ associated with both sides and using $(7.11), (7.12)$ and (7.13),

we can derive

$$
\begin{aligned}
&\int_{\overline{M}}||H||_B^{1-\alpha}(\psi_\delta)_B^{\beta-1}\mathrm{Tr}((A_{\mathcal{H}})^3)_B d\bar{v}-\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||A_{\mathcal{H}}||_B^4 d\bar{v}\\
&\leq\int_{\overline{M}}||H||_B^{1-\alpha}(\psi_\delta)_B^{\beta-1}(\mathrm{Tr}((\mathcal{A}_\xi^\phi)^2\circ A_{\mathcal{H}}))_B d\bar{v}\\
&\quad-\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}(\mathrm{Tr}(\mathcal{A}_\xi^\phi)^2|_{\mathcal{H}})_B(||A_{\mathcal{H}}||^2)_B d\bar{v}\\
&\quad-\frac{\beta-1}{2}\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar{v}\\
&\quad-(\alpha\beta-\alpha+1)\int_{\overline{M}}||H||_B^{-1}(\psi_\delta)_B^{\beta-1}g_{\mathcal{H}}(d||H||,d\psi_\delta)_B d\bar{v}\\
&\quad-\alpha\int_{\overline{M}}||H||_B^{-\alpha-1}(\psi_\delta)_B^{\beta-1}g_{\mathcal{H}}((d||H||\otimes d||H||,h_{\mathcal{H}}-(||H||/n)g_{\mathcal{H}})_B d\bar{v}\\
&\quad+(\beta-1)\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-2}g_{\mathcal{H}}((d||H||\otimes d\psi_\delta,h_{\mathcal{H}}-(||H||/n)g_{\mathcal{H}})_B d\bar{v}\\
&\quad+(1-1/n)\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||\,d||H||\,||_B^2 d\bar{v}\\
&\quad-3\int_{\overline{M}}||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}\mathrm{Tr}_{g_{\mathcal{H}}}(\mathcal{A}_\xi^\phi\circ\mathcal{A}^\phi_{\mathrm{grad}||H||})_B d\bar{v}\\
&\quad-\alpha\int_{\overline{M}}||H||_B^{-2}(\psi_\delta)_B^{\beta}||\,d||H||\,||_B^2 d\bar{v}\\
&\quad-\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}(\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet,\bullet))_B d\bar{v}.
\end{aligned}
\tag{7.14}
$$

Denote by $*_1$ the sum of the first term, the second one, the eight one and the last one in the right-hand side of (7.14), and $*_2$ the sum of the remained terms in the right-hand side of (7.14). Then, by simple calculations, we can derive

$$
\begin{aligned}
*_1\leq\,&2\sqrt{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}||(\mathcal{A}_\xi^\phi)^2|_{\mathcal{H}}||_B\cdot||A_{\mathcal{H}}||_B^2\\
&+3\sqrt{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}||\mathcal{A}_\xi^\phi\circ\mathcal{A}^\phi_{\mathrm{grad}||H||}||_B d\bar{v}\\
&-\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\left(\mathrm{Tr}^\bullet_{g_{\mathcal{H}}}\mathcal{R}(A_{\mathcal{H}}\bullet,\bullet)\right)_B d\bar{v},
\end{aligned}
\tag{7.15}
$$

where we use $-\mathrm{Tr}((\mathcal{A}_\xi^\phi)^2|_{\mathcal{H}})\leq\sqrt{n}||(\mathcal{A}_\xi^\phi)^2|_{\mathcal{H}}||$ and $||H||\leq\sqrt{n||A_{\mathcal{H}}||^2}$. Also, by

simple calculations, we can derive

$$
\begin{aligned}
*_2 \leq{}& \alpha \int_{\overline{M}} ||H||_B^{-\alpha/2-1}(\psi_\delta)_B^{\beta-1/2}||d||H||\,||_B^2 d\bar{v} \\
&+(\alpha\beta+\alpha-1)\int_{\overline{M}} ||H||_B^{-1}\cdot(\psi_\delta)_B^{\beta-1}\cdot||d||H||\,||_B\cdot||d\psi_\delta||_B d\bar{v} \\
&+(\beta-1)\int_{\overline{M}} ||H||_B^{-\alpha/2}(\psi_\delta)_B^{\beta-3/2}||\,d||H||\,||_B\cdot||\,d\psi_\delta||_B d\bar{v} \\
&+\frac{n-1}{n}\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||\,d||H||\,\,||_B^2 d\bar{v},
\end{aligned} \tag{7.16}
$$

where we use $||\,d||H||\otimes d||H||\,|| = ||\,d||H||\,||^2$, $||\,d||H||\otimes d\psi_\delta|| = ||\,d||H||\,||\cdot||\,d\psi_\delta||$ and $||h_{\mathcal{H}} - \frac{||H||}{n}g_{\mathcal{H}}||^2 = \psi_\delta||H||^\alpha$. By noticing $ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$ for any $a, b, \eta > 0$ and $\psi_\delta \leq ||H||^\delta$ $(0 < \delta < 1)$, we have

$$
\begin{aligned}
&\int_{\overline{M}} ||H||_B^{-1}\cdot(\psi_\delta)_B^{\beta-1}\cdot||d||H||\,||_B\cdot||d\psi_\delta||_B d\bar{v} \\
={}& \int_{\overline{M}} (||H||_B^{-\alpha/2}(\psi_\delta)_B^{(\beta-1)/2}||d||H||\,||_B)\cdot(||H||_B^{\alpha/2-1}(\psi_\delta)_B^{(\beta-1)/2}||d\psi_\delta||_B)d\bar{v} \\
\leq{}& \frac{\eta}{2}\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||d||H||\,||_B^2\,d\bar{v} + \frac{1}{2\eta}\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar{v}
\end{aligned} \tag{7.17}
$$

and

$$
\begin{aligned}
&\int_{\overline{M}} ||H||_B^{-\alpha/2}(\psi_\delta)_B^{\beta-3/2}||d||H||\,||_B\cdot||d\psi_\delta||_B d\bar{v} \\
={}& \int_{\overline{M}} \left(||H||_B^{-\alpha/2}(\psi_\delta)_B^{(\beta-1)/2}||d||H||\,||_B\right)\left((\psi_\delta)_B^{(\beta-2)/2}||d\psi_\delta||_B\right)d\bar{v} \\
\leq{}& \frac{\eta}{2}\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||d||H||\,||_B^2\,d\bar{v} + \frac{1}{2\eta}\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar{v}.
\end{aligned} \tag{7.18}
$$

From (7.4) and (7.14) − (7.18), we can derive

$$
\begin{aligned}
&n\varepsilon^2\int_{\overline{M}} ||H||_B^2(\psi_\delta)_B^\beta\,d\bar{v} \\
\leq{}& \frac{(\alpha\beta+\alpha+\beta-2)\eta}{2}\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||d||H||\,||_B^2\,d\bar{v} \\
&+\frac{\alpha\beta+\alpha+\beta-2}{2\eta}\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar{v} \\
&+\left(\alpha+\frac{n-1}{n}\right)\int_{\overline{M}} ||H||_B^{-\alpha}(\psi_\delta)_B^{\beta-1}||d||H||\,||_B^2\,d\bar{v} \\
&+2\sqrt{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\cdot||(\mathcal{A}_\xi^\phi)^2|_{\mathcal{H}}||_B\cdot||A_{\mathcal{H}}||_B^2\,d\bar{v} \\
&+3\sqrt{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\cdot||\mathcal{A}_\xi^\phi\circ\mathcal{A}_{\mathrm{grad}||H||}^\phi||_B\,d\bar{v} \\
&-\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}_{g_{\mathcal{H}}}^{\bullet}\mathcal{R}(A_{\mathcal{H}}\bullet,\bullet)_B\,d\bar{v}.
\end{aligned} \tag{7.19}
$$

Since $0 \leq \delta \leq \frac{1}{2}$ (hence $\frac{2}{3} \leq \alpha \leq 2$), we can derive the desired inequality.

q.e.d.

Also, we can derive the following inequality.

Lemma 7.1.8. *Assume that* $0 \leq \delta \leq \frac{1}{2}$. *Then, for any* $\beta \geq 100\varepsilon^{-2}$, *we have*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\overline{M}}(\psi_\delta)_B^\beta d\bar{v} + 2\int_{\overline{M}}(\psi_\delta)_B^\beta ||H||_B^2 d\bar{v}\\
&+\frac{\beta(\beta-1)}{2}\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2 d\bar{v}\\
&+\frac{\beta\varepsilon^2}{8}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\cdot||d||H||\,||_B^2 d\bar{v}\\
\leq\ &\beta\delta\int_{\overline{M}}(\psi_\delta)_B^\beta ||H||_B^2 d\bar{v} + 16\beta\varepsilon^{-2}K\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}d\bar{v}\\
&-3\beta\delta\int_{\overline{M}}(\psi_\delta)_B^\beta \mathrm{Tr}((\mathcal{A}_\xi^\phi)^2)_{\mathcal{H}})_B d\bar{v} - 6\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^\delta \mathrm{Tr}((\mathcal{A}_\xi^\phi)^2\circ A_{\mathcal{H}})_B d\bar{v}\\
&+\frac{6\beta}{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^\delta \mathrm{Tr}((\mathcal{A}_\xi^\phi)^2)_B d\bar{v}\\
&-4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(((\nabla_\bullet\mathcal{A})_\bullet\circ A_{\mathcal{H}})\cdot,\cdot)_B d\bar{v}\\
&-4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ A_{\mathcal{H}})\cdot,\mathcal{A}_\bullet\cdot)_B d\bar{v}\\
&+4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ\mathcal{A}_\bullet)\cdot,A_{\mathcal{H}}\cdot)_B d\bar{v}.
\end{aligned}
$$

Proof. By multiplying $\beta\psi_\delta^{\beta-1}$ to both sides of the inequality in Lemma 7.1.4 and

integrating over $\overline{M}$, we obtain

$$
\begin{aligned}
&\int_{\overline{M}}\left(\frac{\partial\psi_\delta^\beta}{\partial t}\right)_B d\bar v+\beta(\beta-1)\int_{\overline{M}}(\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2d\bar v\\
&+\frac{\beta\varepsilon^2}{4}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\cdot||d||H||\,||_B^2d\bar v\\
\le\ &\beta\delta\int_{\overline{M}}(\psi_\delta)_B^\beta||H||_B^2d\bar v+2\beta(\alpha-1)\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-1}\cdot||\,d||H||\,||_B\cdot||d\psi_\delta||_Bd\bar v\\
&+16\beta\varepsilon^{-2}K\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}d\bar v\\
&-3\beta\delta\int_{\overline{M}}(\psi_\delta)_B^\beta\mathrm{Tr}((\mathcal{A}_\xi^\phi)^2)_{\mathcal{H}})_Bd\bar v-6\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^\delta\mathrm{Tr}((\mathcal{A}_\xi^\phi)^2\circ A_{\mathcal{H}})_Bd\bar v\\
&+\frac{6\beta}{n}\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^\delta\mathrm{Tr}((\mathcal{A}_\xi^\phi)^2)_Bd\bar v\\
&-4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h(((\nabla_\bullet\mathcal{A})_\bullet\circ A_{\mathcal{H}})\cdot,\cdot)_Bd\bar v\\
&-4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ A_{\mathcal{H}})\cdot,\mathcal{A}_\bullet\cdot)_Bd\bar v\\
&+4\beta\int_{\overline{M}}(\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}\mathrm{Tr}^{\cdot}_{g_{\mathcal{H}}}\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}h((\mathcal{A}_\bullet\circ\mathcal{A}_\bullet)\cdot,A_{\mathcal{H}}\cdot)_Bd\bar v,
\end{aligned}
$$

where we use $\int_{\overline{M}}\triangle_{\mathcal{H}}(\psi_\delta^\beta)_Bd\bar v=0$ and $||A_{\mathcal{H}}||^2\le||H||^2$. From this inequality, $\frac{\partial}{\partial t}(d\bar v)=-2||H||_B^2d\bar v$, $||\,d||H||\,||\cdot||d\psi_\delta||\le\frac{\beta-1}{4||H||^{1-\alpha}}||d\psi_\delta||^2+\frac{||H||^{1-\alpha}}{\beta-1}||\,d||H||\,||^2$, $\alpha\le 2$, $||A_{\mathcal{H}}||^2\le||H||^2$, $\psi_\delta\le||H||^\delta$ and $\beta-1\ge100\varepsilon^{-2}-1\ge16\varepsilon^{-2}$ (which holds because of $\varepsilon\le1$), we can derive the desired inequality. q.e.d.

For a function $\overline{\rho}$ over $\overline{M}\times[0,T)$, denote by $||\overline{\rho}(\cdot,t)||_{L^\beta,\bar g_t}$ the L^β-norm of with respect to $\bar g_t$ and $||\overline{\rho}||_{L^\beta,\bar g}$ the function over $[0,T)$ defined by assigning $||\overline{\rho}(\cdot,t)||_{L^\beta,\bar g_t}$ to each $t\in[0,T)$.

By using Lemmas 7.1.7 and 7.1.8, we can derive the fact.

Lemma 7.1.9. *There exists a positive constant C depending only on K,L and f such that, for any δ and β satisfying*

$$
(7.20)\quad 0\le\delta\le\min\left\{\frac12,\frac{n\varepsilon^2\eta}{3},\frac{n\varepsilon^4}{24(\eta+1)}\right\}\ \text{ and }\ \beta\ge\max\left\{100\varepsilon^{-2},\frac{n\varepsilon^2\eta}{n\varepsilon^2\eta-3\delta}\right\},
$$

the following inequality holds:

$$\sup_{t\in[0,T)}||(\psi_\delta)_B(\cdot,t)||_{L^\beta,\bar g_t}<C.$$

Proof. Set

$$C_1 := (\mathrm{Vol}_{g_0}(M)+1) \sup_{\delta\in[0,1/2]} \max_{M} \psi_\delta(\cdot,0).$$

Then we have $||\psi_\delta(\cdot,0)_B||_{L^\beta,g_0} \leq C_1$. By using the inequalities in Lemmas 7.1.7 and 7.1.8, $||A_{\mathcal{H}}||^2 \leq ||H||^2$ and the Young's inequality, we can show that

$$\begin{aligned} &\frac{\partial}{\partial t}\left(||(\psi_\delta)_B||^\beta_{L^\beta,\bar{g}}\right) \\ \leq\ &\frac{\beta((3\delta - n\varepsilon^2\eta)\beta + n\varepsilon^2\eta)}{2n\varepsilon^2\eta}\int_{\overline{M}} (\psi_\delta)_B^{\beta-2}||d\psi_\delta||_B^2\, d\bar{v} \\ &+\frac{\beta(12\eta\delta\beta + 24\delta - n\varepsilon^4)}{8n\varepsilon^2}\int_{\overline{M}} (\psi_\delta)_B^{\beta-1}||H||_B^{-\alpha}||\, d||H||\ ||^2 d\bar{v} \\ &+C_2||(\psi_\delta)_B||^\beta_{L^\beta,\bar{g}} + C_3 \end{aligned} \tag{7.21}$$

holds for some positive constants C_2 and C_3 depending only on K and L. Hence we can derive

$$\begin{aligned} &\sup_{t\in[0,T)} ||(\psi_\delta)_B(\cdot,t)||_{L^\beta,\bar{g}_t} \\ \leq\ &\left(\left(\frac{C_3}{C_2} + ||(\psi_\delta)_B(\cdot,0)||^\beta_{L^\beta,\bar{g}_0}\right)e^{C_2T} - \frac{C_3}{C_2}\right)^{1/\beta} \\ \leq\ &\left(\left(\frac{C_3}{C_2} + C_1\right)e^{C_2T} - \frac{C_3}{C_2}\right)^{1/\beta}. \end{aligned}$$

q.e.d.

By using this lemma, we can derive the following inequality.

Lemma 7.1.10. *Take any positive constant k. Assume that*

$$0 \leq \delta \leq \min\left\{\frac{1}{2} - \frac{k}{\beta}, \frac{n\varepsilon^2\eta}{3} - \frac{k}{\beta}, \frac{n\varepsilon^4}{24(\eta+1)} - \frac{k}{\beta}\right\} \tag{7.22}$$

and

$$\beta \geq \max\left\{100\varepsilon^{-2}, \frac{n\varepsilon^2\eta}{n\varepsilon^2\eta - 3\delta}\right\}. \tag{7.23}$$

Then the following inequality holds:

$$\sup_{t\in[0,T)}\left(\int_{\overline{M}} ||H_t||_B^k (\psi_\delta(\cdot,t))_B^\beta d\bar{v}\right)^{1/\beta} \leq C,$$

where C is as in Lemma 7.1.9.

Proof. Set $\delta' := \delta + \frac{k}{\beta}$. Clearly we have $||H_t||_B^k(\psi_\delta(\cdot,t))_B^\beta = \psi_{\delta'}^\beta$. From the assumption for δ and β, δ' satisfies (7.20). Hence, from Lemma 7.1.9, we have

$$\left(\int_{\overline{M}} ||H_t||_B^k(\psi_\delta(\cdot,t))_B^\beta d\bar{v}\right)^{1/\beta} = \left(\int_{\overline{M}} (\psi_{\delta'}(\cdot,t))_B^\beta d\bar{v}\right)^{1/\beta} \leq C.$$

q.e.d.

By using the Sobolev's inequality by Hoffman and Spruck ([HS]), we can derive the following inequality.

Lemma 7.1.11. *Let D be a closed domain in M, $\bar{K}$ be the maximal sectional curvature of (N, g_N), $\bar{R}((\bar{f}_t \circ \phi_M)(D))$ the injective radius of (N, g_N) restricted to $(\bar{f}_t \circ \phi_M)(D)$ and ω_n the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Set $b := \sqrt{\bar{K}}$. For any G-invariant non-negative C^1-function ψ on D satisfying*

$$b^2(1-\alpha)^{-2/n}\left(\omega_n^{-1}\cdot \mathrm{Vol}_{\bar{g}_t}(\phi_M(\operatorname{supp}\psi))\right)^{2/n} \leq 1 \tag{7.24}$$

and

$$b^{-1}\sin^{-1} b\cdot(1-\alpha)^{-1/n}\left(\omega_n^{-1}\cdot \mathrm{Vol}_{\bar{g}_t}(\phi_M(\operatorname{supp}\psi))\right)^{1/n} \leq \frac{1}{2}\bar{R}((\bar{f}_t \circ \phi_M)(D)) \tag{7.25}$$

$(0 < \alpha < 1)$, the following inequality holds:

$$\left(\int_{\phi_M(D)} \psi_B^{\frac{n}{n-1}} d\bar{v}\right)^{\frac{n-1}{n}} \leq C(n)\int_{\phi_M(D)} (||d\psi||_B + \psi_B||H||_B)\, d\bar{v},$$

where

$$C(n) := \frac{\pi}{2}\cdot 2^{n-2}\alpha^{-1}(1-\alpha)^{-1/n}\frac{n}{n-1}\omega_n^{-1/n}.$$

By using Lemmas 7.1.9, 7.1.10 and 7.1.11, we shall prove the statement of Proposition 7.1.

Proof of Proposition 7.1. (Step I) First we shall show $T < \infty$. According to Lemma 3.8, we have

$$\frac{\partial ||H||}{\partial t} \geq \triangle_{\mathcal{H}}||H|| + \frac{||H||^3}{n}.$$

Let ρ be the solution of the ordinary differential equation $\frac{dy}{\partial t} = \frac{1}{n}y^3$ with the initial condition $y(0) = \min_M ||H_0||$. This solution ρ is given by

$$\rho(t) = \frac{\min_M ||H_0||}{\sqrt{1-(2/n)\min_M ||H_0||^2 \cdot t}}.$$

We regard ρ as a function over $M \times [0,T)$. Then we have

$$\frac{\partial(||H||-\rho)}{\partial t} \geq \triangle_{\mathcal{H}}(||H||-\rho) + \frac{||H||^3-\rho^3}{n}.$$

Furthermore, by the maximum principle, we can derive that $||H|| \geq \rho$ holds over $M \times [0,T)$. Therefore we obtain

$$||H|| \geq \frac{\min_M ||H_0||}{\sqrt{1-(2/n)\min_M ||H_0||^2 \cdot t}}.$$

This implies that $T \leq \frac{1}{(2/n)\min_M ||H_0||^2} (< \infty)$.

(Step II) Take positive constants δ and β satisfying (7.22) and (7.23). Define a function $\psi_{\delta,k}$ by $\psi_{\delta,k} := \max\{0, \psi_\delta(\cdot,t) - k\}$, where k is any positive number with $k \geq \sup_M \psi_\delta(\cdot,0)$. Set $A_t(k) := \{\phi(u) \mid \psi_\delta(u,t) \geq k\}$ and $\tilde{A}(k) := \bigcup_{t\in[0,T)} (A_t(k) \times \{t\})$, which is finite because of $T < \infty$. For a function $\bar{\rho}$ over $\overline{M} \times [0,T)$, denote by $\int_{A(k)} \bar{\rho}\, d\bar{v}$ the function over $[0,T)$ defined by assigning $\int_{A_t(k)} \bar{\rho}(\cdot,t)\, d\bar{v}_t$ to each $t \in [0,T)$. By multiplying the inequality in Lemma 7.1.4 by $\beta\psi_{\delta,k}^{\beta-1}$, we can show that the inequality in Lemma 7.1.8 holds for $\psi_{\delta,k}$ instead of ψ_δ. From the inequality, the following inequality is derived directly:

$$\frac{\partial}{\partial t}\int_{A(k)} (\psi_{\delta,k})_B^\beta d\bar{v} + \frac{\beta(\beta-1)}{2}\int_{A(k)} (\psi_{\delta,k})_B^{\beta-2} ||d\psi_{\delta,k}||_B^2 d\bar{v} \leq \beta\delta \int_{A(k)} (\psi_{\delta,k})_B^\beta ||H||_B^2 d\bar{v}.$$

Set $\hat{\psi} := \psi_{\delta,k}^{\beta/2}$. On $A_t(k)$, we have

$$\frac{\beta(\beta-1)}{2}(\psi_{\delta,k})_B^{\beta-2}(\cdot,t)||d(\psi_{\delta,k})_B(\cdot,t)||^2 \geq ||d\hat{\psi}_B(\cdot,t)||^2$$

and hence

$$\frac{\partial}{\partial t}\int_{A(k)} \hat{\psi}_B^2 d\bar{v} + \int_{A(k)} ||d\hat{\psi}_B||^2\, d\bar{v} \leq \beta\delta \int_{A(k)} \hat{\psi}_B^2 ||H||_B^2 d\bar{v}.$$

By integrating both sides of this inequality from 0 to any $t_0(\in [0,T))$, we have

$$\int_{A_{t_0}(k)} \hat{\psi}_B^2(\cdot,t_0)d\bar{v}_{t_0} + \int_0^{t_0}\left(\int_{A(k)} ||d\hat{\psi}_B||^2\, d\bar{v}\right)dt \leq \beta\delta \int_0^{t_0}\left(\int_{A(k)} \hat{\psi}_B^2||H||_B^2 d\bar{v}\right)dt.$$

By the arbitrariness of t_0, we have

$$\begin{aligned} &\sup_{t\in[0,T)} \int_{A_t(k)} \hat{\psi}_B^2(\cdot,t)d\bar{v}_t + \int_0^T\left(\int_{A(k)} ||d\hat{\psi}_B||^2\, d\bar{v}\right)dt \\ &\leq \beta\delta \int_0^T\left(\int_{A(k)} \hat{\psi}_B^2||H||_B^2 d\bar{v}\right)dt, \end{aligned} \tag{7.26}$$

where we use $k \geq \sup_M \psi_\delta(\cdot,0)$. From $k \geq \sup_M \psi_\delta(\cdot,0)$, we have $A_0(k) = \emptyset$. Since f satisfies the conditions $(*_1^\alpha)$ and $(*_2^\alpha)$, so is also f_t $(0 \leq t < T)$ because $\mathrm{Vol}_{\bar{g}_t}(\overline{M})$ decreases with respect to t by Lemma 3.11. Hence we can apply the Sobolev's inequality in Lemma 7.1.11 to f_t $(0 \leq t < T)$. By using the Sobolev's inequality in Lemma 7.1.11 and the Hölder's inequality, we can derive

$$\begin{aligned} &\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{\frac{n-1}{n}} \\ &\leq C(n)\left(\int_{\overline{M}} ||d(\hat{\psi}^{\frac{2(n-1)}{n-2}})(\cdot,t)||_B\, d\bar{v}_t + \int_{\overline{M}} \hat{\psi}_B^{\frac{2(n-1)}{n-2}}(\cdot,t)\cdot||H_t||_B\, d\bar{v}_t\right) \\ &= C(n)\left(\frac{2(n-1)}{n-2}\int_{\overline{M}} \hat{\psi}_B^{\frac{n}{n-2}}(\cdot,t)\cdot||d\hat{\psi}(\cdot,t)||_B\, d\bar{v}_t + \int_{\overline{M}} \hat{\psi}_B^{\frac{2(n-1)}{n-2}}(\cdot,t)\cdot||H_t||_B\, d\bar{v}_t\right) \\ &\leq C(n)\left\{\frac{2(n-1)}{n-2}\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{1/2}\cdot\left(\int_{\overline{M}} ||d\hat{\psi}(\cdot,t)||_B^2\, d\bar{v}_t\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{\frac{n-1}{n}}\cdot\left(\int_{\overline{M}} ||H_t||_B^n\, d\bar{v}_t\right)^{1/n}\right\}. \end{aligned}$$

Also, since $\psi_\delta(\cdot,t) \geq k$ on $A_t(k)$, it follows from Lemma 7.1.10 that

$$\left(\int_{\overline{M}} ||H_t||_B\, d\bar{v}_t\right)^{1/n} \leq k^{-\beta/n}\left(\int_{\overline{M}} ||H_t||_B \psi_\delta^\beta\, d\bar{v}_t\right)^{1/n} \leq k^{-\beta/n}\cdot C^{\beta/n},$$

where C is as in Lemma 7.1.9. Hence we obtain

$$\begin{aligned} &\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{\frac{n-1}{n}} \\ &\leq C(n)\left\{\frac{2(n-1)}{n-2}\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{1/2}\cdot\left(\int_{\overline{M}} ||d\hat{\psi}(\cdot,t)||_B^2\, d\bar{v}_t\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\, d\bar{v}_t\right)^{\frac{n-1}{n}}\cdot k^{-\beta/n}\cdot C^{\beta/n}\right\}, \end{aligned}$$

that is,

$$
\begin{aligned}
&\left(\int_{\overline{M}} ||d\hat{\psi}(\cdot,t)||_B^2\,d\bar{v}_t\right)^{1/2}\\
&\geq \frac{n-2}{2C(n)(n-1)}\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\,d\bar{v}_t\right)^{\frac{n-2}{2n}}\left(1-C(n)\cdot\left(\frac{C}{k}\right)^{\beta/n}\right).
\end{aligned}
$$

Set

$$
k_1 := \max\left\{\sup_M \psi_\delta(\cdot,0),\ C(n)^{n/\beta}\cdot C\right\}.
$$

Assume that $k \geq k_1$. Then we have

$$
\begin{aligned}
&\int_{\overline{M}} ||d\hat{\psi}(\cdot,t)||_B^2\,d\bar{v}_t\\
&\geq \left(\frac{n-2}{2C(n)(n-1)}\right)^2\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\,d\bar{v}_t\right)^{\frac{n-2}{n}}\left(1-C(n)\cdot\left(\frac{C}{k}\right)^{\beta/n}\right)^2.
\end{aligned}
\tag{7.27}
$$

From (7.26) and (7.27), we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_{\overline{M}} \hat{\psi}_B^2(\cdot,t)d\bar{v}_t + \hat{C}(n,k)\int_0^T\left(\int_{\overline{M}} \hat{\psi}_B^{\frac{2n}{n-2}}(\cdot,t)\,d\bar{v}_t\right)^{\frac{n-2}{n}}dt\\
&\leq \beta\delta\int_0^T\left(\int_{\overline{M}} \hat{\psi}_B^2||H_t||_B^2 d\bar{v}_t\right)dt,
\end{aligned}
\tag{7.28}
$$

where $\hat{C}(n,k) := \left(\frac{(n-2)(1-C(n)\cdot(C/k)^{\beta/n})}{2C(n)(n-1)}\right)^2$. Set

$$
q := \begin{cases} \dfrac{n}{n-2} & (n\geq 3)\\ \text{any positive number} & (n=2)\end{cases}
$$

and $q_0 := 2 - 1/q$ and

$$
||A_t(k)||_T := \int_0^T\left(\int_{A_t(k)} d\bar{v}_t\right)dt.
$$

By using the interpolation inequality, we can derive

$$
\left(\int_{\overline{M}} \hat{\psi}_B^{2q_0}\,d\bar{v}_t\right)^{1/q_0} \leq \left(\int_{\overline{M}} \hat{\psi}_B^2\,d\bar{v}_t\right)^{1-1/q_0}\cdot\left(\int_{\overline{M}} \hat{\psi}_B^{2q}\,d\bar{v}_t\right)^{1/qq_0}.
$$

By using this inequality and the Young inequality, we can derive

$$
\begin{aligned}
&\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2q_0}(\cdot,t)\,d\bar{v}_t\right)dt\right)^{1/q_0}\\
&\leq\left(\int_0^T\left(\left(\int_{\overline{M}}\hat{\psi}_B^{2}(\cdot,t)\,d\bar{v}_t\right)^{q_0-1}\cdot\left(\int_{\overline{M}}\hat{\psi}_B^{2q}(\cdot,t)\,d\bar{v}_t\right)^{1/q}\right)dt\right)^{1/q_0}\\
&\leq\left(\sup_{t\in[0,T]}\int_{\overline{M}}\hat{\psi}_B^{2}(\cdot,t)\,d\bar{v}_t\right)^{\frac{q_0-1}{q_0}}\cdot\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2q}(\cdot,t)\,d\bar{v}_t\right)^{1/q}dt\right)^{1/q_0}\\
&\leq\sup_{t\in[0,T]}\int_{\overline{M}}\hat{\psi}_B^{2}(\cdot,t)\,d\bar{v}_t+\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2q}(\cdot,t)\,d\bar{v}_t\right)^{1/q}dt.
\end{aligned}
\tag{7.29}
$$

We may assume that $\hat{C}(n,k)<1$ holds by replacing $C(n)$ to a bigger positive number and furthemore k to a positive number bigger such that $1-C(n)\cdot\left(\frac{C}{k}\right)^{\beta/n}>0$ holds for the replaced number $C(n)$. Then, from (7.28) and (7.29), we obtain

$$
\begin{aligned}
&\hat{C}(n,k)\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2q_0}(\cdot,t)\,d\bar{v}_t\right)dt\right)^{1/q_0}\\
&\leq\beta\delta\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2}||H_t||_B^2d\bar{v}_t\right)dt.
\end{aligned}
\tag{7.30}
$$

On the other hand, by using the Hölder's inequality, we obtain

$$
\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2}||H_t||_B^2d\bar{v}_t\right)dt\leq||A_t(k)||_T^{\frac{r-1}{r}}\cdot\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2r}||H_t||_B^{2r}d\bar{v}_t\right)dt\right)^{1/r},
$$

where r is any positive constant with $r>1$. From (7.30) and this inequality, we obtain

$$
\begin{aligned}
&\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2q_0}(\cdot,t)\,d\bar{v}_t\right)dt\right)^{1/q_0}\\
&\leq\hat{C}(n,k)^{-1}\beta\delta||A_t(k)||_T^{\frac{r-1}{r}}\cdot\left(\int_0^T\left(\int_{\overline{M}}\hat{\psi}_B^{2r}||H_t||_B^{2r}d\bar{v}_t\right)dt\right)^{1/r}.
\end{aligned}
\tag{7.31}
$$

On the other hand, according to Lemma 7.1.10, we have

$$
\int_{\overline{M}}\widehat{\psi}_B^{2r}||H_t||_B^{2r}\,d\bar{v}_t\leq C^{2r}
\tag{7.32}
$$

for some positive constant C (depending only on K, L and f) by replacing r to a bigger positive number if necessary. Also, by using the Hölder inequality, we obtain

$$\begin{aligned}&\int_0^T \left(\int_{\overline{M}} \hat{\psi}_B^2(\cdot,t)\,d\bar{v}_t\right)dt\\ \leq\ &||A_t(k)||_T^{\frac{q_0-1}{q_0}} \cdot \left(\int_0^T \left(\int_{\overline{M}} \hat{\psi}_B^{2q_0}(\cdot,t)\,d\bar{v}_t\right)dt\right)^{1/q_0}.\end{aligned}$$

From (7.31), (7.32) and this inequality, we obtain

$$\int_0^T \left(\int_{\overline{M}} \hat{\psi}_B^2(\cdot,t)\,d\bar{v}_t\right)dt \leq ||A_t(k)||_T^{2-1/q_0-1/r} \cdot C^2 \cdot \hat{C}(n,k)^{-1}\beta\delta. \tag{7.33}$$

We may assume that $2 - 1/q_0 - 1/r > 1$ holds by replacing r to a bigger positive number if necessary. Take any positive constants h and k with $h > k \geq k_1$. Then we have

$$\begin{aligned}&\int_0^T \left(\int_{\overline{M}} \psi_{\delta,k}^{\beta} d\overline{v}_t\right)dt \geq \int_0^T \left(\int_{\overline{M}} (\psi_{\delta,k} - \psi_{\delta,h})^{\beta} d\overline{v}_t\right)dt\\ \geq\ &\int_0^T \left(\int_{A_t(h)} |h-k|^{\beta}\,d\overline{v}_t\right)dt = |h-k|^{\beta} \cdot ||A_t(h)||_T.\end{aligned}$$

From this inequality and (7.33), we obtain

$$|h-k|^{\beta} \cdot ||A_t(h)||_T \leq ||A_t(k)||_T^{2-1/q_0-1/r} \cdot C^2 \cdot \hat{C}(n,k)^{-1}\beta\delta. \tag{7.34}$$

Since $\bullet \mapsto ||A_t(\bullet)||_T$ is a non-increasing and non-negative function and (7.34) holds for any $h > k \geq k_1$, it follows from the Stambaccha's iteration lemma that $||A_t(k_1 + d)||_T = 0$, where d is a positive constant depending only on $\beta, \delta, q_0, r, C, \widehat{C}(n,k)$ and $||A_t(k_1)||_T$. This implies that $\sup_{t\in[0,T)} \max_{\overline{M}} \psi_\delta(\cdot,t) \leq k_1 + d < \infty$. This completes the proof. q.e.d.

8 Estimate of the gradient of the mean curvature from above

In this section, we shall derive the following estimate of grad$||H||$ from above by using Proposition 7.1.

Proposition 8.1. *For any positive constant b, there exists a constant $C(b, f_0)$ (depending only on b and f_0) satisfying*

$$||\mathrm{grad}\,||H||\,||^2 \leq b \cdot ||H||^4 + C(b, f_0) \quad \text{on} \quad M \times [0,T).$$

We prepare some lemmas to prove this proposition.

Lemma 8.1.1. *The family* $\{||\mathrm{grad}_t||H_t||\,||^2\}_{t\in[0,T)}$ *satisfies the following equation:*

$$
\begin{aligned}
&\frac{\partial||\mathrm{grad}\,||H||\,||^2}{\partial t}-\triangle_{\mathcal H}(||\mathrm{grad}\,||H||\,||^2)\\
&=-2||\nabla^{\mathcal H}\mathrm{grad}\,||H||\,||^2+2||A_{\mathcal H}||^2\cdot||\mathrm{grad}\,||H||\,||^2\\
&\quad+2||H||\cdot g_{\mathcal H}(\mathrm{grad}(||A_{\mathcal H}||^2),\mathrm{grad}\,||H||)\\
&\quad+2g_{\mathcal H}((A_{\mathcal H})^2(\mathrm{grad}\,||H||),\mathrm{grad}\,||H||)\\
&\quad-6||H||\cdot g_{\mathcal H}(\mathrm{grad}(\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal H},\mathrm{grad}\,||H||)\\
&\quad-6\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal H}\cdot||\mathrm{grad}\,||H||\,||^2.
\end{aligned}
\tag{8.1}
$$

Hence we have the following inequality:

$$
\begin{aligned}
&\frac{\partial||\mathrm{grad}\,||H||\,||^2}{\partial t}-\triangle_{\mathcal H}(||\mathrm{grad}\,||H||\,||^2)\\
&\leq-2||\nabla^{\mathcal H}\mathrm{grad}\,||H||\,||^2+4||A_{\mathcal H}||^2\cdot||\mathrm{grad}\,||H||\,||^2\\
&\quad+2||H||\cdot g_{\mathcal H}(\mathrm{grad}(||A_{\mathcal H}||^2),\mathrm{grad}\,||H||)\\
&\quad+6||H||\cdot||\mathrm{grad}(\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal H})||\cdot||\mathrm{grad}\,||H||\,||\\
&\quad-6\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal H}\cdot||\mathrm{grad}\,||H||\,||^2.
\end{aligned}
\tag{8.2}
$$

Proof. By using Lemmas 3.1 and 3.8, we have

$$
\begin{aligned}
&\frac{\partial||\mathrm{grad}\,||H||\,||^2}{\partial t}=\frac{\partial g_{\mathcal H}}{\partial t}(\mathrm{grad}\,||H||,\mathrm{grad}\,||H||)+2g_{\mathcal H}\left(\mathrm{grad}\left(\frac{\partial||H||}{\partial t}\right),\mathrm{grad}\,||H||\right)\\
&=-2||H||\cdot h_{\mathcal H}(\mathrm{grad}\,||H||,\mathrm{grad}\,||H||)+2g_{\mathcal H}(\mathrm{grad}(\triangle_{\mathcal H}||H||),\ \mathrm{grad}\,||H||)\\
&\quad+2g_{\mathcal H}(\mathrm{grad}(||H||\cdot||A_{\mathcal H}||^2)),\ \mathrm{grad}\,||H||)\\
&\quad-6g_{\mathcal H}(\mathrm{grad}(||H||\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal H},\ \mathrm{grad}\,||H||).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
\triangle_{\mathcal H}(||\mathrm{grad}\,||H||\,||^2)=&\,2g_{\mathcal H}(\triangle^{\mathcal H}_{\mathcal H}(\mathrm{grad}\,||H||),\ \mathrm{grad}\,||H||)\\
&+2g_{\mathcal H}(\nabla^{\mathcal H}\mathrm{grad}\,||H||,\ \nabla^{\mathcal H}\mathrm{grad}\,||H||)
\end{aligned}
$$

and

$$\triangle^{\mathcal H}_{\mathcal H}(\mathrm{grad}\,||H||)=\mathrm{grad}(\triangle_{\mathcal H}||H||)+||H||\cdot A_{\mathcal H}(\mathrm{grad}\,||H||)-(A_{\mathcal H})^2(\mathrm{grad}\,||H||).$$

By using these relations and noticing $g_{\mathcal H}(A_{\mathcal H}(\bullet),\cdot)=-h_{\mathcal H}(\bullet,\cdot)$, we can derive the desired evolution equation (8.1). The inequality (8.2) is derived from (8.1) and

$$g_{\mathcal H}((A_{\mathcal H})^2(\mathrm{grad}\,||H||),\ \mathrm{grad}\,||H||)\leq||A_{\mathcal H}||^2\cdot||\mathrm{grad}\,||H||\,||^2.$$

q.e.d.

Lemma 8.1.2. *The family* $\left\{\frac{||\mathrm{grad}_t||H_t||\,||^2}{||H_t||}\right\}_{t\in[0,T)}$ *satisfies the following inequality:*

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{||\mathrm{grad}\,||H||\,||^2}{||H||}\right)-\triangle_{\mathcal{H}}\left(\frac{||\mathrm{grad}\,||H||\,||^2}{||H||}\right)\\
&\leq\frac{3||\mathrm{grad}\,||H||\,||^2}{||H||}\cdot||A_{\mathcal{H}}||^2+2g_{\mathcal{H}}(\mathrm{grad}(||A_{\mathcal{H}}||^2,\,\mathrm{grad}\,||H||)\\
&\quad+6||\mathrm{grad}(\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}||\cdot||\mathrm{grad}\,||H||\,||-\frac{3}{||H||}\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\cdot||\mathrm{grad}\,||H||\,||^2
\end{aligned}\tag{8.3}$$

Proof. By a simple calculation, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{||\mathrm{grad}\,||H||\,||^2}{||H||}\right)-\triangle_{\mathcal{H}}\left(\frac{||\mathrm{grad}\,||H||\,||^2}{||H||}\right)\\
=&\frac{1}{||H||}\left(\frac{\partial||\mathrm{grad}\,||H||\,||^2}{\partial t}-\triangle_{\mathcal{H}}(||\mathrm{grad}\,||H||\,||^2)\right)\\
&-\frac{||\mathrm{grad}\,||H|^{|}\,||^2}{||H||^2}\left(\frac{\partial||H||}{\partial t}-\triangle_{\mathcal{H}}||H||\right)\\
&+\frac{2}{||H||^2}g_{\mathcal{H}}(\mathrm{grad}\,||H||,\,\mathrm{grad}(||\mathrm{grad}\,||H||\,||^2)).
\end{aligned}$$

From this relation, Lemmas 3.8 and (8.2), we can derive the desired inequality.

q.e.d.

From Lemma 3.8, we can derive the following evolution equation directly.

Lemma 8.1.3. *The family* $\left\{||H_t||^3\right\}_{t\in[0,T)}$ *satisfies the following evolution equation:*

$$\begin{aligned}
&\frac{\partial||H||^3}{\partial t}-\triangle_{\mathcal{H}}(||H||^3)\\
&=3||H||^3\cdot||A_{\mathcal{H}}||^2-6||H||\cdot||\mathrm{grad}\,||H||\,||^2-9||H||^3\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}.
\end{aligned}$$

By using Lemmas 3.8, 3.10 and Proposition 7.1, we can derive the following evolution inequality.

Lemma 8.1.4. *The family* $\left\{\left(||(A_{\mathcal{H}})_t||^2-\frac{||H_t||^2}{n}\right)\cdot||H_t||\right\}_{t\in[0,T)}$ *satisfies the following evolution inequality:*

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\right)-\triangle_{\mathcal{H}}\left(\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\right)\\
\leq&-\frac{2(n-1)}{3n}||H||\cdot||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2+\check{C}(n,C_0,\delta)\cdot||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2\\
&+3||H||\cdot||A_{\mathcal{H}}||^2\cdot\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\\
&-2||H||\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\cdot\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\\
&-4||H||^2\cdot\mathrm{Tr}\left((\mathcal{A}^{\phi}_{\xi})^2)\circ\left(A_{\mathcal{H}}-\frac{||H||}{n}\cdot\mathrm{id}\right)\right)\\
&-2||H||\cdot\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}\left(\left(A_{\mathcal{H}}-\frac{||H||}{n}\cdot\mathrm{id}\right)(\bullet),\bullet\right)\\
&-3\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}.
\end{aligned}$$

Proof. By using Lemmas 3.8 and 3.10, we can derive

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\right)\\
&-\triangle_{\mathcal{H}}\left(\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\right)\\
=&\frac{2||H||}{n}\cdot||\mathrm{grad}\,||H||\,||^2+2||H||\cdot||A_{\mathcal{H}}||^2\cdot\left(\mathrm{Tr}(((A_{\mathcal{H}})^2)-\frac{||H||^2}{n}\right)\\
&-2||H||\cdot||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2+||H||\cdot||A_{\mathcal{H}}||^2\cdot\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\\
&-g_{\mathcal{H}}\left(\mathrm{grad}\left(||A_{\mathcal{H}}||^2-\frac{||H||}{n}\cdot\mathrm{id}\right),\,\mathrm{grad}\,||H||\right)\\
&-2||H||\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\cdot\left(\mathrm{Tr}(((A_{\mathcal{H}})^2)-\frac{||H||^2}{n}\right)\\
&-4||H||^2\cdot\mathrm{Tr}\left((\mathcal{A}^{\phi}_{\xi})^2)\circ\left(A_{\mathcal{H}}-\frac{||H||}{n}\cdot\mathrm{id}\right)\right)\\
&-2||H||\cdot\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}\left(\left(A_{\mathcal{H}}-\frac{||H||}{n}\cdot\mathrm{id}\right)(\bullet),\bullet\right)\\
&-3\left(||A_{\mathcal{H}}||^2-\frac{||H||^2}{n}\right)\cdot||H||\cdot\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}.
\end{aligned}\tag{8.4}$$

On the othe hand, by using $||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n} = ||A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id}\,||$, we can derive

$$
\begin{aligned}
&\left| g_{\mathcal{H}} \left(\mathrm{grad} \left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n} \right), \, \mathrm{grad}\,||H|| \right) \right| \\
= &\left| d \left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n} \right) (\mathrm{grad}\,||H||) \right| \\
= &\, 2 \left| g_{\mathcal{H}} \left(\nabla^{\mathcal{H}}_{\mathrm{grad}||H||} \left(A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id} \right), \, A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id} \right) \right| \\
\leq &\, 2||\mathrm{grad}||H||\,|| \cdot ||\nabla^{\mathcal{H}} A_{\mathcal{H}}|| \cdot \left|\left| A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id} \right|\right| \\
\leq &\, 2n||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2 \cdot \left|\left| A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id} \right|\right|,
\end{aligned}
$$

where we use $\frac{1}{n}||\mathrm{grad}||H||\,||^2 \leq ||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2$. Also, according to Proposition 7.1, we have

$$\left|\left| A_{\mathcal{H}} - \frac{||H||}{n} \cdot \mathrm{id} \right|\right| \leq \sqrt{C_0} \cdot ||H||^{1-\delta/2}.$$

Hence we have

$$
\begin{aligned}
&\left| g_{\mathcal{H}} \left(\mathrm{grad} \left(||A_{\mathcal{H}}||^2 - \frac{||H||^2}{n} \right), \, \mathrm{grad}\,||H|| \right) \right| \\
\leq &\, 2n\sqrt{C_0}||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2 \cdot ||H||^{1-\delta/2}.
\end{aligned} \tag{8.5}
$$

Furthermore, according to the Young's inequality:

$$ab \leq \varepsilon \cdot a^p + \varepsilon^{-1/(p-1)} \cdot b^q \quad (\forall\, a > 0, \; b > 0) \tag{8.6}$$

(where p and q are any positive constants with $\frac{1}{p} + \frac{1}{q} = 1$ and ε is any positive constant), we have

$$2n\sqrt{C_0}||H||^{1-\delta/2} \leq \frac{2(n-1)}{3n} \cdot ||H|| + \breve{C}(n, C_0, \delta), \tag{8.7}$$

where $\breve{C}(n, C_0, \delta)$ is a positive constant only on n, C_0 and δ. Also, we have

$$||\nabla^{\mathcal{H}} A_{\mathcal{H}}||^2 \geq \frac{3}{n+2}||\nabla^{\mathcal{H}} H||^2.$$

From (8.4) and these inequalities, we can derive the desired evolution inequality.

q.e.d.

By using Lemmas 3.9, 8.1.2, 8.1.3 and 8.1.4, we shall prove Theorem Proposition 8.1.

Proof of Proposition 8.1. Define a function ρ over $M\times[0,T)$ by

$$\rho := \frac{||\mathrm{grad}\,||H||\,||^2}{||H||}+C_1||H||\left(||A_{\mathcal H}||^2-\frac{||H||^2}{n}\right)+C_1\cdot\breve{C}(n,C_0,\delta)||A_{\mathcal H}||^2-b||H||^3,$$

where b is any positive constant and C_1 is a positive constant which is sufficiently big compared to n and b. By using Lemmas 3.9, 8.1.2, 8.1.3 and 8.1.4, we can derive

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}-\triangle_{\mathcal H}\rho\\
&\leq \frac{3||\mathrm{grad}\,||H||\,||^2}{||H||}\cdot||A_{\mathcal H}||^2+2g_{\mathcal H}(\mathrm{grad}(||A_{\mathcal H}||^2),\,\mathrm{grad}\,||H||)\\
&\quad-\frac{2(n-1)}{3n}\cdot C_1\cdot||H||\cdot||\nabla^{\mathcal H}A_{\mathcal H}||^2\\
&\quad+3C_1\cdot||H||\cdot||A_{\mathcal H}||^2\left(||A_{\mathcal H}||^2-\frac{||H||^2}{n}\right)\\
&\quad+2C_1\cdot\breve{C}(n,C_0,\delta)\cdot||A_{\mathcal H}||^4-3b||H||^3\cdot||A_{\mathcal H}||^2+6b||H||\cdot||\mathrm{grad}\,||H||\,||^2\\
&\quad+6||\mathrm{grad}\,||H||\,||\cdot||\mathrm{grad}(\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H})||\\
&\quad-\frac{3}{||H||}\cdot||\mathrm{grad}\,||H||\,||^2\cdot\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\\
&\quad-2C_1||H||\cdot\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\cdot\left(||A_{\mathcal H}||^2-\frac{||H||^2}{n}\right)\\
&\quad-4C_1||H||^2\cdot\mathrm{Tr}\left((\mathcal{A}^\phi_\xi)^2)\circ\left(A_{\mathcal H}-\frac{||H||}{n}\cdot\mathrm{id}\right)\right)\\
&\quad-2C_1||H||\cdot\mathrm{Tr}^\bullet_{g_{\mathcal H}}\mathcal{R}\left(\left(A_{\mathcal H}-\frac{||H||}{n}\cdot\mathrm{id}\right)(\bullet),\bullet\right)\\
&\quad-3C_1\left(||A_{\mathcal H}||^2-\frac{||H||^2}{n}\right)\cdot||H||\cdot\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\\
&\quad-2C_1\cdot\breve{C}(n,C_0,\delta)||A_{\mathcal H}||^2\cdot\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\\
&\quad-4C_1\cdot\breve{C}(n,C_0,\delta)||H||\cdot\mathrm{Tr}\left(((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}\circ A_{\mathcal H}\right)\\
&\quad-2C_1\cdot\breve{C}(n,C_0,\delta)\mathrm{Tr}^\bullet_{g_{\mathcal H}}\mathcal{R}(A_{\mathcal H}\bullet,\bullet)+9b\cdot||H||^3\cdot\mathrm{Tr}((\mathcal{A}^\phi_\xi)^2)_{\mathcal H}.
\end{aligned}\tag{8.8}$$

Also, in similar to (8.5), we obtain

$$\begin{aligned}&|g_{\mathcal H}(\mathrm{grad}(||A_{\mathcal H}||^2),\,\mathrm{grad}\,||H||)|\\ &\leq 2n\sqrt{C_0}||\nabla^{\mathcal H}A_{\mathcal H}||^2\cdot||H||^{1-\delta/2}.\end{aligned}$$

This implies together with (8.7) that

$$|g_{\mathcal H}(\mathrm{grad}(||A_{\mathcal H}||^2),\,\mathrm{grad}\,||H||)|\leq\left(\frac{2(n-1)}{3n}\cdot||H||+\breve{C}(n,C_0,\delta)\right)||\nabla^{\mathcal H}A_{\mathcal H}||^2.\tag{8.9}$$

Denote by T^1V the unit tangent bundle of V. Define a function Ψ over T^1V by

$$\Psi(X) := ||\, d(\mathrm{Tr}(\mathcal{A}^{\phi}_X)^2)_{\widetilde{\mathcal{H}}})\,|| \quad (X \in T^1V).$$

It is clear that Ψ is continuous. Set $\widehat{K}_1 := \sup_{t\in[0,T)} \max_M ||\mathrm{grad}(\mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}})||$, which is finite because Ψ is continuous and the closure of $\underset{t\in[0,T)}{\cup}\phi(f_t(M))$ is compact. Also, we have

$$\mathrm{Tr}^{\bullet}_{g_{\mathcal{H}}}\mathcal{R}\left(\left(A_{\mathcal{H}} - \frac{||H||}{n}\cdot \mathrm{id}\right)(\bullet), \bullet\right) \le \widehat{K}_2 \cdot \left|\left|A_{\mathcal{H}} - \frac{||H||}{n}\cdot \mathrm{id}\right|\right| \tag{8.10}$$

for some positive constant $\widehat{K}_2$ because of the homogeneity of N. By using (8.7), (8.9), (8.10), $||A_{\mathcal{H}}|| \le ||H||$, $\frac{1}{n}||\mathrm{grad}||H||\,||^2 \le ||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2$ and Proposition 7.1, we can derive

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} - \triangle_{\mathcal{H}}\rho \\
&\le \left(3n + \frac{4(n-1)}{3n} - \frac{2(n-1)C_1}{3n} + 6nb\right)||H||\cdot||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2 \\
&\quad +2\breve{C}(n,C_0,\delta)\cdot||\nabla^{\mathcal{H}}A_{\mathcal{H}}||^2 + 3C_0\cdot C_1||H||^{5-\delta} + 2C_1\cdot\breve{C}(n,C_0,\delta)||H||^4 \\
&\quad -3b||H||^5 + 6\widehat{K}_1||\mathrm{grad}\,||H||\,|| + \frac{3\widehat{K}_1}{||H||}\cdot||\mathrm{grad}\,||H||\,||^2 \\
&\quad +2C_0\cdot C_1\cdot\widehat{K}_1||H||^{3-\delta} + 4C_1\cdot\sqrt{C_0}\cdot\widehat{K}_1||H||^{3-\delta/2} \\
&\quad +2C_1\cdot\widehat{K}_2\cdot\sqrt{C_0}\cdot||H||^{2-\delta/2} + 3C_0\cdot C_1\cdot\widehat{K}_1\cdot||H||^{3-\delta} \\
&\quad +2C_1\cdot\breve{C}(n,C_0,\delta)\cdot\widehat{K}_1\cdot||H||^2 + 4C_1\cdot\breve{C}(n,C_0,\delta)\cdot\widehat{K}_1\cdot||H||^3 \\
&\quad +2C_1\cdot\breve{C}(n,C_0,\delta)\cdot\widehat{K}_2\cdot||H|| + 9b\cdot\widehat{K}_1\cdot||H||^3.
\end{aligned} \tag{8.11}$$

Furthermore, by using the Young's inequality (8.6) and the fact that C_1 is sufficiently big compared to n and b, we can derive that

$$\frac{\partial\rho}{\partial t} - \triangle_{\mathcal{H}}\rho \le C_3(n,C_0,C_1,b,\delta,\widehat{K}_1,\widehat{K}_2)$$

holds for some positive constant $C_3(n,C_0,C_1,b,\delta,\widehat{K}_1,\widehat{K}_2)$ only on $n, C_0, C_1, b, \delta, \widehat{K}_1$ and $\widehat{K}_2$. This together with $T < \infty$ implies that

$$\begin{aligned}
\max_M \rho_t &\le \max_M \rho_0 + C_3(n,C_0,C_1,b,\delta,\widehat{K}_1,\widehat{K}_2)t \\
&\le \max_M \rho_0 + C_3(n,C_0,C_1,b,\delta,\widehat{K}_1,\widehat{K}_2)\cdot T
\end{aligned}$$

$(0 \le t < T)$. Therefore, we obtain

$$||\mathrm{grad}\,||H||\,||^2 \le b||H||^4 + \max_M \rho_0\cdot||H|| + C_3(n,C_0,C_1,b,\delta,\widehat{K}_1,\widehat{K}_2)\cdot T\cdot||H||.$$

Furthermore, by using the Young inequality (8.6), we obtain

$$||\mathrm{grad}\,||H||\,||^2 \leq 2b||H||^4 + C_4(n, C_0, C_1, b, \delta, \widehat{K}_1, \widehat{K}_2, T)$$

holds for some positive constant $C_4(n, C_0, C_1, b, \delta, \widehat{K}_1, \widehat{K}_2, T)$ only on $n, C_0, C_1, b, \delta, \widehat{K}_1$ $\widehat{K}_2$ and T. Since b is any positive constant and $C_4(n, C_0, C_1, b, \delta, \widehat{K}_1, \widehat{K}_2, T)$ essentially depends only on n and f_0, we obtain the statement of Proposition 8.1.

q.e.d.

9 Proof of Theorem 6.1.

In this section, we shall prove Theorem 6.1. G. Huisken ([Hu2]) obtained the evolution inequality for the squared norm of all iterated covariant derivatives of the shape operators of the mean curvature flow in a complete Riemannian manifold satisfying curvature-pinching conditions in Theorem 1.1 of [Hu2]. See the proof of Lemma 7.2 (Page 478) of [Hu2] about this evolution inequality. In similar to this evolution inequality, we obtain the following evolution inequality.

Lemma 9.1. *For any positive integer m, the family $\{||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||^2\}_{t\in[0,T)}$ satisfies the following evolution inequality:*

$$\begin{aligned}
&\frac{\partial ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||^2}{\partial t} - \triangle_{\mathcal{H}}||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||^2 \\
&\leq -2||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||^2 + C_4(n,m) \\
&\quad\times\left(\sum_{i+j+k=m}||(\nabla^{\mathcal{H}})^i A_{\mathcal{H}}||\cdot||(\nabla^{\mathcal{H}})^j A_{\mathcal{H}}||\cdot||(\nabla^{\mathcal{H}})^k A_{\mathcal{H}}||\cdot||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||\right. \\
&\qquad\left.+C_5(m)\sum_{i\leq m}||(\nabla^{\mathcal{H}})^i A_{\mathcal{H}}||\cdot||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}|| + C_6(m)||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||\right),
\end{aligned} \tag{9.1}$$

where $C_4(n,m)$ is a positive constant depending only on n, m and $C_i(m)$ $(i = 5, 6)$ are positive constants depending only on m.

In similar to Corollary 12.6 of [Ha], we can derive the following interpolation inequality.

Lemma 9.2. *Let S be an element of $\Gamma(\pi_M^*(T^{(1,1)}M))$ such that, for any $t \in [0,T)$, S_t is a G-invariant $(1,1)$-tensor field on M. For any positive integer m, the following*

inequality holds:

$$\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^i S_{\mathcal{H}})||_B^{2m/i}\, d\bar{v} \le C(n,m)\cdot \max_M ||S_{\mathcal{H}}||^{2(m/i-1)}\cdot \int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m S_{\mathcal{H}})||_B^2\, d\bar{v},$$

where $C(n,m)$ is a positive constant depending only on n and m.

From these lemmas, we can derive the following inequality.

Lemma 9.3. *For any positive integer m, the following inequality holds:*

$$\begin{aligned} &\frac{d}{dt}\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v} + 2\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^{m+1} A_{\mathcal{H}}||_B^2\, d\bar{v} \\ &\le C_7(n,m,C_6(m),\mathrm{Vol}(M_0))\cdot\left(\max_M ||A_{\mathcal{H}}||^2+1\right) \\ &\quad\times\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v} + \left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v}\right)^{1/2}\right), \end{aligned} \tag{9.2}$$

where $C_7(n,m,C_6(m),\mathrm{Vol}(M_0))$ is a positive constant depending only on $n,m,C_6(m)$ and the volume $\mathrm{Vol}(M_0)$ of $M_0 = f_0(M)$.

Proof. By using (9.1) and the generalized Hölder inequality, we can derive

$$\begin{aligned} &\frac{d}{dt}\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v} + 2\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^{m+1} A_{\mathcal{H}}||_B^2\, d\bar{v} \\ &\le C_4(n,m)\cdot\left(\sum_{i+j+k=m}\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^i A_{\mathcal{H}}||_B^{\frac{2m}{i}}\, d\bar{v}\right)^{\frac{i}{2m}}\cdot\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^j A_{\mathcal{H}}||_B^{\frac{2m}{j}}\, d\bar{v}\right)^{\frac{j}{2m}} \\ &\qquad\times\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^k A_{\mathcal{H}}||_B^{\frac{2m}{k}}\, d\bar{v}\right)^{\frac{k}{2m}}\cdot\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v}\right)^{\frac12} \\ &\quad +C(n,m)\widetilde{C}(m)\sum_{i\le m}\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^i A_{\mathcal{H}}||_B^{\frac{2m}{i}}\, d\bar{v}\right)^{\frac{i}{2m}}\cdot\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^{\frac{2m}{2m-i}}\, d\bar{v}\right)^{\frac{2m-i}{2m}} \\ &\quad +C(n,m)\widetilde{C}(m+1)\cdot\left(\int_{\overline{M}} ||(\nabla^{\mathcal{H}})^m A_{\mathcal{H}}||_B^2\, d\bar{v}\right)^{\frac12}\cdot\left(\int_{\overline{M}} d\bar{v}\right)^{\frac12}. \end{aligned}$$

From this inequality and Lemma 9.2, we can derive the desired inequality. q.e.d.

From this lemma, we can derive the following statement.

Proposition 9.4. *The family $\{||A_{\mathcal{H}}||^2\}_{t\in[0,T)}$ is not uniform bounded.*

Proof. Suppose that $\sup_{t\in[0,T)}\max_M ||A_{\mathcal H}||^2<\infty$. Denote by C_A this supremum. Define a function Φ over $[0,T)$ by

$$\Phi(t):=\int_{\overline{M}}||(\nabla^{\mathcal H})^m(A_{\mathcal H})_t||_B^2\,d\bar v_t\qquad (t\in[0,T)).$$

Then, according to (9.2), we have

$$\frac{d\Phi}{dt}\le C_7(n,m,C_6(m),\mathrm{Vol}(M_0))\cdot(C_A+1)\cdot(\Phi+\Phi^{1/2}).$$

Assume that $\sup_{t\in[0,T)}\Phi>1$. Set $E:=\{t\in[0,T)\,|\,\Phi(t)>1\}$. Take any $t_0\in E$. Then $\Phi\ge 1$ holds over $[t_0,t_0+\varepsilon)$ for some a sufficiently small positive number ε. Hence we have

$$\frac{d\Phi}{dt}\le 2C_7(n,m,C_6(m),\mathrm{Vol}(M_0))\cdot(C_A+1)\cdot\Phi$$

on $[t_0,t_0+\varepsilon)$. From this inequality, we can derive

$$\Phi(t)\le\Phi(t_0)e^{2C_7(n,m,C_6(m),\mathrm{Vol}(M_0))\cdot(C_A+1)(t-t_0)}\qquad (t\in[t_0,t_0+\varepsilon))$$

and hence

$$\Phi(t)\le\Phi(t_0)e^{2C_7(n,m,C_6(m),\mathrm{Vol}(M_0))\cdot(C_A+1)T}\qquad (t\in[t_0,t_0+\varepsilon)).$$

This fact together with the arbitrariness of t_0 implies that Φ_t is uniform bounded. Thus, we see that

$$\sup_{t\in[0,T)}\int_{\overline{M}}||(\nabla^{\mathcal H})^m(A_{\mathcal H})_t||_B^2\,d\bar v_t<\infty$$

holds in general. Furthermore, since this inequality holds for any positive integar m, it follows from Lemma 9.2 that

$$\sup_{t\in[0,T)}\int_{\overline{M}}||(\nabla^{\mathcal H})^m(A_{\mathcal H})_t||_B^l\,d\bar v_t<\infty$$

holds for any positive integar m and any positive constant l. Hence, by the Sobolev's embedding theorem, we obtain

$$\sup_{t\in[0,T)}\max_M||(\nabla^{\mathcal H})^m(A_{\mathcal H})_t||<\infty.$$

Since this fact holds for any positive integer m, f_t converges to a C^∞-embedding f_T as $t\to T$ in C^∞-topology. This implies that the mean curvature flow f_t extends after T because of the short time existence of the mean curvature flow starting from f_T. This contradicts the definition of T. Therefore we obtain

$$\sup_{t\in[0,T)}\max_M||A_{\mathcal H}||^2=\infty.$$

q.e.d.

By imitating the proof of Theorem 4.1 of [A1,2], we can show the folowing fact, where we note that more general curvature flows (including mean curvature flows as special case) is treated in [A1,2].

Lemma 9.5. *The following uniform boundedness holds:*

$$\inf_{t\in[0,T)} \max\{\varepsilon > 0 \,|\, (A_{\mathcal{H}})_t \geq \varepsilon ||H_t|| \cdot \mathrm{id} \text{ on } M\} > 0$$

and hence

$$\sup_{(x,t)\in M\times[0,T)} \frac{\lambda_{\max}(x,t)}{\lambda_{\min}(x,t)} \leq \frac{1}{\varepsilon_0},$$

where $\lambda_{\max}(x,t)$ (resp. $\lambda_{\min}(x,t)$) denotes the maximum (resp. minimum) eigenvalue of $(A_{\mathcal{H}})_{(x,t)}$ and ε_0 denotes the above infimum.

Proof. Since

$$\begin{aligned}&\left(\frac{\partial h_{\mathcal{H}}}{\partial t} - \triangle^{\mathcal{H}}_{\mathcal{H}} h_{\mathcal{H}}\right)(X,Y)\\ &= -2||H|| \cdot h_{\mathcal{H}}(A_{\mathcal{H}}(X),Y) + g_{\mathcal{H}}\left(\left(\frac{\partial A_{\mathcal{H}}}{\partial t} - \triangle^{\mathcal{H}}_{\mathcal{H}} A_{\mathcal{H}}\right)(X),Y\right).\end{aligned}$$

From this relation, Lemmas 3.5 and 3.8, we can derive

$$\begin{aligned}&\frac{\partial A_{\mathcal{H}}}{\partial t} - \triangle^{\mathcal{H}}_{\mathcal{H}} A_{\mathcal{H}}\\ &= -2||H||((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} + \mathrm{Tr}\left((A_{\mathcal{H}})^2 - ((\mathcal{A}^{\phi})^2)_{\mathcal{H}}\right) \cdot A_{\mathcal{H}} - \mathcal{R}^{\sharp}.\end{aligned}$$

Furthermore, from this evolution equation and Lemma 3.8, we can derive

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{A_{\mathcal{H}}}{||H||}\right) - \triangle^{\mathcal{H}}_{\mathcal{H}}\left(\frac{A_{\mathcal{H}}}{||H||}\right)\\ &= \frac{1}{||H||}\nabla^{\mathcal{H}}_{\mathrm{grad}\,||H||}\left(\frac{A_{\mathcal{H}}}{||H||}\right) + \frac{||\mathrm{grad}\,||H||\,||^3}{||H||^3} \cdot A_{\mathcal{H}} - 2((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}\\ &\quad + \frac{2}{||H||} \cdot \mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \cdot A_{\mathcal{H}} - \frac{1}{||H||}\mathcal{R}^{\sharp}.\end{aligned}$$

For simplicity, we set

$$S_{\mathcal{H}} := g_{\mathcal{H}}\left(\frac{1}{||H||}A_{\mathcal{H}}(\bullet), \bullet\right)$$

and

$$P(S)_{\mathcal{H}} := \frac{||\mathrm{grad}\,||H||\,||^3}{||H||^3} \cdot h_{\mathcal{H}} - 2(((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}})_{\flat}$$
$$+\frac{2}{||H||} \cdot \mathrm{Tr}((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}} \cdot h_{\mathcal{H}} - \frac{1}{||H||}\mathcal{R},$$

where $(((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}})_{\flat}$ is defined by $(((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}})_{\flat}(\bullet.\bullet) := g_{\mathcal{H}}(((\mathcal{A}^{\phi}_{\xi})^2)_{\mathcal{H}}(\bullet), \bullet)$. Also, set

$$\varepsilon_0 := \max\{\varepsilon > 0 \,|\, (S_{\mathcal{H}})_0 \geq \varepsilon g_{\mathcal{H}}\}.$$

Then, for any $(x,t) \in M \times [0,T)$, any $\varepsilon > 0$ and any $X \in \mathrm{Ker}(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)}$, we can show $P(S_{\mathcal{H}} + \varepsilon g_{\mathcal{H}})_{(x,t)}(X,X) \geq 0$. Hence, by Theorem 4.1 (the maximum principle), we can derive that $(S_{\mathcal{H}})_t \geq \varepsilon_0 g_{\mathcal{H}}$, that is, $(A_{\mathcal{H}})_t \geq \varepsilon_0 ||H_t|| g_{\mathcal{H}}$ holds for all $t \in [0,T)$. From this fact, it follows that $\lambda_{\min}(x,t) \geq \varepsilon_0 ||H_{(x,t)}||$ holds for all $(x,t) \in M \times [0,T)$. Hence we obtain

$$\sup_{(x,t)\in M\times[0,T)} \frac{\lambda_{\max}(x,t)}{\lambda_{\min}(x,t)} \leq \sup_{(x,t)\in M\times[0,T)} \frac{\lambda_{\max}(x,t)}{\varepsilon_0 ||H_{(x,t)}||} \leq \frac{1}{\varepsilon_0}.$$

q.e.d.

According to this lemma, we see that such a case as in Figure 3 does not happen.

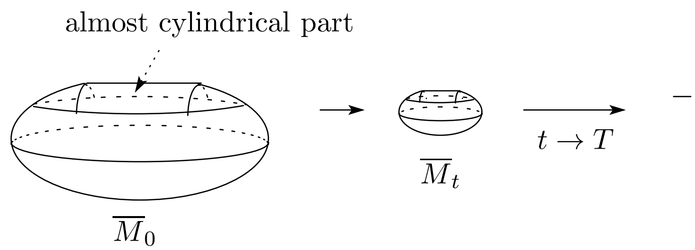


Figure 3.

By using Proposition 9.4 and Lemma 9.5, we shall prove the statement (i) of Theorem 6.1.

Proof of (i) of Theorem 6.1. According to Proposition 9.4 and Lemma 9.5, we have

$$\lim_{t\to T} \min_{x\in M} \lambda_{\min}(x,t) = \infty.$$

Set $\Lambda_{\min}(t) := \min_{x\in M} \lambda_{\min}(x,t)$. Let $x_{\min}(t)$ be a point of $\overline{M}$ with $\lambda_{\min}(x_{\min}(t),t) = \Lambda_{\min}(t)$ and set $\bar{x}_{\min}(t) := \phi_M(x_{\min}(t))$. Denote by $\gamma_{\bar{f}_t(\bar{x}_{\min}(t))}$ the normal geodesic

of $\bar{f}_t(\overline{M})$ starting from $\bar{f}_t(\bar{x}_{\min}(t))$. Set $p_t := \gamma_{\bar{f}_t(\bar{x}_{\min}(t))}(1/\Lambda_{\min}(t))$. Since N is of non-negative curvature, the focal radii of $\overline{M}_t$ along any normal geodesic are smaller than or equal to $\frac{1}{\Lambda_{\min}(t)}$. This implies that $\bar{f}_t(\overline{M})$ is included by the geodesic sphere of radius $\frac{1}{\Lambda_{\min}(t)}$ centered at p_t in N. Hence, since $\lim_{t\to T}\frac{1}{\Lambda_{\min}(t)}=0$, we see that, as $t\to T$, $\overline{M}_t$ collapses to a one-point set, that is, M_t collapses to a G-orbit.

q.e.d.

Denote by $(\mathrm{Ric}_{\overline{M}})_t$ the Ricci tensor of $\overline{g}_t$ and let $\mathrm{Ric}_{\overline{M}}$ be the element of $\Gamma(\pi_{\overline{M}}^*(T^{(0,2)}\overline{M}))$ defined by $(\mathrm{Ric}_{\overline{M}})_t$'s. To show the statement (ii) of Theorem 6.1, we prepare the following some lemmas.

Lemma 9.6. (i) *For the section* $\mathrm{Ric}_{\overline{M}}$, *the following relation holds:*

$$\mathrm{Ric}_{\overline{M}}(X,Y)=-3\mathrm{Tr}(\mathcal{A}^{\phi}_{X^L}\circ\mathcal{A}^{\phi}_{Y^L})_{\mathcal{H}}-\overline{g}(\overline{A}^2X,Y)+||\overline{H}||\cdot\overline{g}(\overline{A}X,Y)\tag{9.3}$$

($X,Y\in\Gamma(\pi_{\overline{M}}^*(T\overline{M}))$), where where X^L (resp. Y^L) is the horizontal lift of X (resp. Y) to V.

(ii) *Let λ_1 be the smallest eigenvalue of $\overline{A}_{(x,t)}$. Then we have*

$$(\mathrm{Ric}_{\overline{M}})_{(x,t)}(v,v)\geq(n-1)\lambda_1^2\overline{g}_{(x,t)}(v,v)\qquad(v\in T_x\overline{M}).\tag{9.4}$$

Proof. Denote by $\overline{\mathrm{Ric}}$ the Ricci tensor of N. By the Gauss equation, we have

$$\mathrm{Ric}_{\overline{M}}(X,Y)=\overline{\mathrm{Ric}}(X,Y)-\overline{g}(\overline{A}^2X,Y)+||\overline{H}||\overline{g}(\overline{A}X,Y)-\overline{R}(\xi,X,Y,\xi)$$
$$(X,Y\in T\overline{M}).$$

Also, by a simple calculation, we have

$$\overline{\mathrm{Ric}}(X,Y)=-3\mathrm{Tr}(\mathcal{A}^{\phi}_{X^L}\circ\mathcal{A}^{\phi}_{Y^L})_{\mathcal{H}}+3g_{\mathcal{H}}((\mathcal{A}^{\phi}_{X^L}\circ\mathcal{A}^{\phi}_{Y^L})(\xi),\,\xi)$$

and

$$\overline{R}(\xi,X,Y,\xi)=3g_{\mathcal{H}}((\mathcal{A}^{\phi}_{X^L}\circ\mathcal{A}^{\phi}_{Y^L})(\xi),\,\xi)$$

$(X,Y\in\Gamma(\pi_{\overline{M}}^*(T\overline{M})))$. From these relations, we obtain the relation (9.3).

Next we show the inequality in the statement (ii). Since $\mathcal{A}^{\phi}_{v^L}$ is skew-symmetric, we have $\mathrm{Tr}((\mathcal{A}^{\phi}_{v^L})^2)\leq 0$. Also we have

$$-\overline{g}_{(x,t)}(\overline{A}^2_{(x,t)}(v),v)+||\overline{H}_{(x,t)}||\cdot\overline{g}_{(x,t)}(\overline{A}_{(x,t)}(v),v)\geq(n-1)\lambda_1^2\cdot\overline{g}_{(x,t)}(v,v).$$

Hence, from the relation in (i), we can derive the inequality (9.4). q.e.d.

Accoriding to the Myers's theorem, we have the following fact even if $(\overline{M}, \overline{g}_t)$ is a Riemannian orbifold.

Lemma 9.7. *Fix $t \in [0, T)$. Assume that $(\mathrm{Ric}_{\overline{M}})_{(x,t)}(v,v) \geq (n-1)K\overline{g}_{(x,t)}(v,v)$ holds for any $x \in \overline{M}$ and any $v \in T_x\overline{M}$, where K is a positive constant. Then the first conjugate radius along any geodesic γ in $(\overline{M}, \overline{g}_t)$ is smaller than or equal to $\frac{\pi}{\sqrt{K}}$.*

By using Propositions 8.1, 9.4 and these lemmas, we prove the statement (ii) of Theorem 6.1.

Proof of (ii) of Theorem 6.1. (Step I) According to Proposition 8.1, for any positive constant b, there exists a constant $C(b, f_0)$ (depending only on b and f_0) satisfying

$$||\mathrm{grad}\, ||H||\, ||^2 \leq b \cdot ||H||^4 + C(b, f_0) \quad \text{on} \quad M \times [0, T).$$

According to Proposition 9.4, we have $\lim_{t \to T} ||H_t||_{\max} = \infty$. Hence there exists a positive constant $t(b)$ with $||H_t||_{\max} \geq \left(\frac{C(b, f_0)}{b}\right)^{1/4}$ for any $t \in [t(b), T)$. Then we have

$$||\mathrm{grad}\, ||H_t||\, || \leq \sqrt{2b}||H_t||^2_{\max} \tag{9.5}$$

for any $t \in [t(b), T)$. Fix $t_0 \in [t(b), T)$. Let x_{t_0} be a maximal point of $||H_{t_0}||$. Take any geodesic γ of length $\frac{1}{\sqrt{2}||H_{t_0}||_{\max} \cdot b^{1/4}}$ starting from x_{t_0}. According to (9.5), we have

$$||H_{t_0}|| \geq (1 - b^{1/4})||H_{t_0}||_{\max}$$

along γ. From the arbitrariness of t_0, this fact holds for any $t \in [t(b), T)$.

(StepII) For any $x \in \overline{M}$, denote by $\gamma_{\overline{f}_t(x)}$ the normal geodesic of $\overline{f}_t(\overline{M})$ starting from $\overline{f}_t(x)$. Set $p_t := \gamma_{\overline{f}_t(x)}(1/\lambda_{\min}(x,t))$ and $q_t(s) := \gamma_{\overline{f}_t(x)}(s/\lambda_{\max}(x,t))$. Since N is of non-negative curvature, the focal radii of $\overline{f}_t(M)$ at x are smaller than or equal to $1/\lambda_{\min}(x,t)$. Denote by $G_2(TN)$ the Grassmann bundle of N of 2-planes and $\mathrm{Sec} : G_2(TN) \to \mathbb{R}$ the function defined by assigning the sectional curvature of Π to each element Π of $G_2(TN)$. Since $\overline{\bigcup_{t \in [0,T)} \overline{f}_t(\overline{M})}$ is compact, there exists the maximum of Sec over $\overline{\bigcup_{t \in [0,T)} \overline{f}_t(\overline{M})}$. Denote by $\kappa_{\max}$ this maximum. It is easy to show that the focal radii of $\overline{f}_t(\overline{M})$ at x are bigger than or equal to $\widehat{c}/\lambda_{\max}(x,t)$

for some positive constant $\widehat{c}$ depending only on $\kappa_{\max}$. Hence a sufficiently small neighborhood of $\overline{f}_t(x)$ in $\overline{f}_t(\overline{M})$ is included by the closed domain surrounded by the geodesic spheres of radius $1/\lambda_{\min}(x,t)$ centered at p_t and that of radius $\widehat{c}/\lambda_{\max}(x,t)$ centered at $q_t(\widehat{c})$. On the other hand, according to Lemma 9.5, we have

$$\sup_{(x,t)\in M\times[0,T)} \frac{\lambda_{\max}(x,t)}{\lambda_{\min}(x,t)} < \infty.$$

By using these facts, we can show

$$\sup_{t\in[0,T)} \frac{||H_t||_{\max}}{||H_t||_{\min}} < \infty$$

and

$$\inf_{t\in[0,T)} \max\{\varepsilon > 0 \,|\, (A_{\mathcal{H}})_t \geq \varepsilon ||H_t|| \cdot \mathrm{id} \text{ on } M\} > 0.$$

Set

$$C_0 := \sup_{t\in[0,T)} \frac{||H_t||_{\max}}{||H_t||_{\min}}$$

and

$$\varepsilon_0 := \inf_{t\in[0,T)} \max\{\varepsilon > 0 \,|\, (A_{\mathcal{H}})_t \geq \varepsilon ||H_t|| \cdot \mathrm{id} \text{ on } M\}.$$

Then, since $A_{\mathcal{H}} \geq \varepsilon_0 ||H||_{\min} \cdot \mathrm{id}$ on $M \times [0,T)$, it follows from (ii) of Lemma 9.6 that

$$(\mathrm{Ric}_{\overline{M}})_{(x,t)}(v,v) \geq (n-1)\varepsilon_0^2 \cdot ||H_t||_{\min}^2 \cdot \overline{g}_{(x,t)}(v,v)$$

for any $(x,t) \in M \times [0,T)$ and any $v \in T_x\overline{M}$. Hence, according to Lemma 9.7, the first conjugate radius along any geodesic γ in $(\overline{M}, \overline{g}_t)$ is smaller than or equal to $\frac{\pi}{\varepsilon_0 ||H_t||_{\min}}$ for any $t \in [0,T)$. This implies that $\exp_{\overline{f}_t(x)}\left(B_{\overline{f}_t(x)}\left(\frac{\pi}{\varepsilon_0||H_t||_{\min}}\right)\right) = \overline{M}$ holds for any $t \in [0,T)$, where $\exp_{f_t(x)}$ denotes the exponential map of $(\overline{M}, \overline{g}_t)$ at $\overline{f}_t(x)$ and $B_{\overline{f}_t(x)}\left(\frac{\pi}{\varepsilon_0||H_t||_{\min}}\right)$ denotes the closed ball of radius $\frac{\pi}{\varepsilon_0||H_t||_{\min}}$ in $T_{\overline{f}_t(x)}\overline{M}$ centered at the zero vector $\mathbf{0}$. By the arbitrariness of b (in (Step I)), we may assume that $b \leq \frac{\varepsilon_0^4}{4\pi^4 C_0^4}$. Then we have

$$\frac{1}{\sqrt{2}||H_t||_{\max} \cdot b^{1/4}} \geq \frac{\pi}{\varepsilon_0 ||H_t||_{\min}}$$

$(t \in [0,T))$. Let t_0 be as in Step I. Then it follows from the above facts that

$$||H_{t_0}|| \geq (1 - b^{1/4})||H_{t_0}||_{\max}$$

holds on $\overline{M}$. From the arbitariness of t_0, it follows that

$$||H|| \geq (1 - b^{1/4})||H||_{\max}$$

holds on $\overline{M} \times [t(b), T)$. In particular, we obtain

$$\frac{||H||_{\max}}{||H||_{\min}} \leq \frac{1}{1 - b^{1/4}}$$

on $\overline{M} \times [t(b), T)$. Therefore, by approaching b to 0, we can derive

$$\lim_{t \to T} \frac{||H_t||_{\max}}{||H_t||_{\min}} = 1.$$

q.e.d.

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